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The Higher Mental Processes in Learning

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A DISSERTATION

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I. INTRODUCTION

The present investigation is an experimental study of the mental processes involved in the solution of certain novel problems and in the utilization of experiences so gained for the subsequent mastery of other problems of a similar nature. The problems were chosen with a view to the possibility of accurate measurement of the progress in their mastery. They are arranged roughly in the order of increasing complexity, and are so related that the solutions for later problems are, for the most part, simply more generalized statements of solutions for earlier ones. In the mastery of these problems all forms of mental process, from very simple acts of perception to fairly difficult and complicated acts of abstraction and generalization, were involved. It is these latter processes chiefly to which attention is directed in the following pages.

In its use of objective methods the study is closely related to the large experimental literature in the field of animal learning and the so-called lower forms of human learning. Its dominant interest in the processes of abstraction and generalization brings it into relation with the researches of Külpe, Mittenzwei, Grünbaum, Moore, Aveling, and Fisher in this field.[1] Its relation to the more dynamic studies of Cleveland[2] and Ruger is

[1] Külpe, Oswald, "Versuche uber Abstraktion," Ber. uber den I Kong. f. exp. Psych. in Geissen, 1904.

Mittenzwei, Kuno, "Uber abstrahierende Apperception," Psych. Stud. 1907, 2.

Grunbaum, A. A., "Uber die Abstraktion der Gleichheit," Arch. f. d. ges. Psychol. 1908, 12.

Moore, Thomas Verner, "The Process of Abstraction: An Experimental Study," University of California Pubs. in Psych., Vol. I.

Aveling, Francis, "The Consciousness of the Universal."

Fisher, Sara Carolyn, "The Process of Generalizing Abstraction," Psych. Rev., Mon. Sup., Vol. XXI, No. 2, 1916. Fisher summarizes the work in generalization and abstraction prior to 1916.

[2] Cleveland, A. A., "The Psychology of Chess and Learning to Play It," Am. Jour. Psych., Vol. 18, 1907.

When a single problem of this sort is solved, the solution is usually couched in more or less specific terms which do not readily function in the subsequent solution of other similar problems. In the solving of a series of related problems, however, it is possible to observe the gradual abstraction of common elements and the association of these elements with appropriate terms, leading finally to the formulation of a general principle for the solution of all problems of the series. If the mastery of 14 as the initial number of beads constitutes the first problem, mastery of 15 as the initial number will constitute the second problem, mastery of 16 as the initial number the third, and so on until enough problems have been solved to permit the subject to develop a general formula for his guidance in drawing from any number of beads.[2]

In the first problem the subject learned that he could not win if required to draw from 3, 6, 9, or 12 beads. He also learned that when the initial number of beads was 14, he *could* win by drawing so as to compel his opponent to draw from these "*critical*" numbers. In the later problems of the series he discovers that these numbers are critical (i.e., the one who must draw from them inevitably loses the game) in all problems of the series, regardless of the initial number of beads presented for solution. Here also he generalizes to the effect that all multiples of 3 are critical numbers, and that he can win any number which is not a multiple of 3 by reducing it to such a multiple at his first draw and through successively lower multiples of 3 to 0 at subsequent draws. Aside, then, from the development of specific responses for the solution of individual problems, the learning process here consists in the formation of (1) a general concept through which the essential elements of all problems may be represented by a single term and treated as a unit, and (2) a

[2] For the sake of uniformity it is necessary to fix upon a definite degree of mastery of problems to be required in all cases. Some difficulty arises here owing to the fact that it is impossible for the subject to win when the initial number of beads presented for solution is a multiple of 3. We have considered such problems "solved" when the subject expressed a conviction that he could not win and called for a new problem. All other problems have been considered solved when two consecutive trials were won.

system of draws under the control of this concept, by means of which all *non-critical* numbers of the series may invariably be won by the subject. The basic concept here developed may be termed the *critical-number* concept. Its functioning and further development through use may be observed by requiring the subject to solve some additional series of problems somewhat similar to those described above.[3]

New series of problems of this sort may be had by varying the numbers of beads which may be taken at a single draw. For example, the extension of the numbers which may be drawn to 1, 2, or 3 (instead of only 1 or 2 as above) yields a new series of problems in which the critical numbers are multiples of 4. Further extension of the numbers which may be drawn to 1, 2, 3, or 4 beads yields another series of problems in which the critical numbers are multiples of 5, etc. By further changes in the numbers which may be drawn an indefinite number of similar series of problems may be obtained. The various series of problems thus obtained will hereafter be designated by the lowest and the highest numbers which may be drawn. Thus *Series 1-2* will denote the series of problems in which only 1 or 2 beads may be taken at a draw; *Series 1-3,* the series in which 1, 2, or 3 beads may be drawn; *Series 2-3,* the series in which only 2 or 3 beads may be drawn, etc.

It will be worth while to direct our attention to the location of critical numbers in the various series since these numbers furnish the key to the solution of the various problems and series of problems. We have observed already that all multiples of 3 are critical numbers in Series 1-2. A clearer insight into the reason why these numbers are critical in this series will facilitate the search for critical numbers in other series of problems. In Series 1-2 any pair of draws may result in the removal of either 2, 3, or 4 beads (the possible combinations of S's and E's draws being 1-1, 1-2, 2-1, 2-2, and the sum of these combinations 2, 3,

[3] A series of problems was considered solved when the subject gave an adequate general formula for the solution of all problems of the series or in some other manner clearly indicated his ability to solve all new problems of the series at sight.

3, and 4 respectively). At any draw either S or E can take away such a number as to make the sum of his own draw and the immediately preceding draw of his opponent equal to 3 by taking 1 when the opponent takes 2, and 2 when the opponent takes 1. Thus either S or E can cause the number of beads to be reduced by 3 with each successive pair of draws, beginning with any draw of his opponent, but he cannot force a reduction by 2's or 4's in this manner for if the opponent's draw is 2, the sum of the pair of draws will necessarily be greater than 2, and if the opponent's draw is 1, the sum of the pair will necessarily be less than 4. The only number, therefore, by which successive reductions can thus be forced is 3, and it is clear that 3 is under control in this manner only because it is the sum of the low and the high draw. If, then, we designate the low draw by the letter L and the high draw by H, we may say in more general terms that *all multiples of the sum of L + H are critical numbers.*[4] Notice that these numbers are critical not merely because they are multiples of the sum of L + H but because they are greater than 0 by exact multiples of this sum, and 0 is critical by definition. If the conditions of the game were so altered as to make 1 instead of 0 a critical number, the critical numbers would be represented by the terms of an arithmetical progression in which the common difference is L + H and the first term is 1.[5]

[4] That this formula holds for all series of problems of this sort wherein winning consists in securing the last draw, can be shown by substituting the general terms, L and H, for 1 and 2 in the foregoing statement. Thus either S or E can take away such a number of beads at any draw as to make the sum of his own draw and the opponent's immediately preceding draw equal to L + H. If one's opponent draws L, one can draw H; if the opponent draws H, one can draw L, and if the opponent draws L + x, one can bring the sum of the pair of draws up to L + H by taking away H — x beads. Moreover, L + H is the only number by which successive reductions can thus regularly be forced for if the opponent's draw is L, one cannot draw enough to raise the sum of the pair of draws above L + H, and if the opponent's draw is H, one cannot draw so low a number as to make the sum of the pair of draws less than L + H.

[5] This condition is actually realized if in Series 1-2, for example, the definition of success is so modified that drawing last constitutes losing instead of winning the game.

In all series of problems in which L is 2 both 0 and 1 may be said to be critical by definition. The critical numbers are here represented by the terms of two arithmetical progressions having each a common difference of L + H. The first terms of these progressions are 0 and 1 respectively. When L is 3, 2 also becomes a critical number. In fact, it will be noticed that all numbers below L, including 0, are in every series critical. These numbers may be called the *basic* critical numbers. Each basic critical number is the first term of an arithmetical progression in which the common difference is L + H and of which all terms are critical numbers. The critical numbers of any series are therefore 0, 1, 2.......L — 1 and all numbers which are greater than any of these basic critical numbers by exactly a multiple of the sum of L + H. This generalization holds for all series in which all numbers between L and H may be drawn. Series of this sort have been called *continuous* series. All series of problems described in the foregoing pages are continuous series. Other series of problems requiring more difficult generalizations may be had by restricting the draws to L and H. Series of this sort in which only L and H may be drawn will be known as *discontinuous* series. Only a limited number of discontinuous series were solved by our subjects, owing to the difficulty of manipulation when the values of L and H are high. To facilitate their analysis these discontinuous series with some of their critical numbers are listed below. Bear in mind that the numbers by which a discontinuous series is indicated are the *only* numbers which may be drawn in that series.

Series (discontinuous).	*Critical Numbers.*
1 or 3;	*0*, *2*, *4*, 6, *8*, 10, *12*, 14, *16*, etc.
1 or 5;	*0*, *2*, 4, *6*, 8, 10, *12*, 14, 16, etc.
1 or 7;	*0*, 2, 4, 6, *8*, 10, 12, 14, *16*, etc.
2 or 6;	*0,1*, 4,5, *8,9*, 12,13, *16,17*, 20,21, etc.
2 or 10;	*0,1*, 4,5, 8,9, *12,13*, 16,17, 20,21, etc.
3 or 9;	*0,1,2*, 6,7,8, *12,13,14*, 18,19,20, *24,25,26*, etc.
4 or 12;	*0,1,2,3*, 8,9,10,11, *16,17,18,19*, 24,25,26,27, etc.
1 or 4;	*0*, 2, *5*, 7, *10*, 12, *15*, 17, *20*, *22*, etc.
1 or 6;	*0*, 2, 4, *7*, 9, 11, *14*, 16, 18, *21*, etc.
1 or 8;	*0*, 2, 4, 6, *9*, 11, 13, 15, *18*, 20, etc.
2 or 5;	*0,1*, 4,–, *7,8*, 11,—, *14,15*, 18,—, etc.

2 or 7; *0,1*, 4, 5, *9,10*, 13,14, *18,19*, 22,23, etc.

2 or 8; *0,1*, 4,5, *10,11*, 14,15, *20,21*, 24, 25, etc.

2 or 9; *0,1*, 4,5, 8,–, *11,12*, 15, 16, 19,—, *22,23*, etc.

3 or 7; *0,1,2*, 6,–,–, *10,11,12*, 16,—,—, *20,21,22*, etc.

3 or 8; *0,1,2*, 6,7,–, *11,12,13*, 17,18,—, *22,23,24*, etc.

4 or 9; *0,1,2,3*, 8,–,–,–, *13,14,15,16*, 21,—,—,—, etc.

The numbers in italics are those obtained by applying our old formula to these series. These numbers will be known as *primary* critical numbers and groups of them as primary C groups. Within the intervals between primary C groups there are now other critical numbers which have become critical because of the restriction of draws to L and H. Such numbers will be known as *secondary* critical numbers. The secondary critical numbers fall into order in accordance with a principle already applied to the primary critical numbers; namely, that there are certain basic critical numbers each of which is the first term of an arithmetical progression in which the common difference is L + H and of which all terms are critical numbers. Our task will therefore be confined to finding the basic secondary critical numbers in the series listed above; i.e., those secondary critical numbers which fall in the interval between the first two groups of primary critical numbers.

Note that in every series in the list *all numbers below H, which may be obtained by adding a multiple of 2L to any number in the primary C group, are critical.* Why are these numbers critical? and why do they not extend beyond H — 1 above 0? Obviously the possibility of using the high draw no longer exists when the number of beads has been reduced below H. Below this point the game must necessarily proceed by successive reductions of L beads at each draw, and each pair of draws will result in the reduction of the number by 2L. Now if, after reduction below H, the number of beads remaining is equal to any number in the primary C group plus a multiple of 2L, it will itself be critical, because the one who draws first in the first pair of draws will necessarily draw first in all succeeding pairs of draws, wherefore the last and winning draw will fall to his opponent. Concerning the second question, it is clear that any number which is greater than 0, or than any other member of

the basic primary C group, by exactly the amount of H, cannot be critical; because *it can be reduced by one draw to less than L,* i.e., to a basic critical number. This is the reason why 4, 6, and 8 are not critical numbers in Series 1 or 4, 1 or 6, and 1 or 8 respectively. This is the reason also why the secondary C groups sometimes consist of less than L critical numbers, as in Series 2 or 5, 2 or 9, 3 or 7, 3 or 8, and 4 or 9. Each of the numbers in these series, which would have been critical but for the limiting effect of the H draw, is indicated in the list by a dash. All numbers that are greater than a basic secondary critical number by an exact multiple of L + H, are themselves critical for the same reason that numbers which are greater than a basic primary critical number by that amount, are critical.

To recapitulate, we have found certain uniformities in the distribution of critical numbers in our series of problems as follows: (1) There is in *every* series a group of critical numbers extending from 0 to L minus 1 inclusive, which we have called the basic primary C group. (2) All numbers in any series, which are greater than any number in the basic primary C group by exactly a multiple of the sum of L + H, are critical. (3) In the discontinuous series all numbers below H which are greater than any number in the basic primary C group by exactly a multiple of 2L, are critical. These are the basic secondary critical numbers. (4) All numbers in a series, which are greater than a basic secondary critical number by exactly a multiple of the sum of L + H, are critical.[6] The primary critical numbers are found in all series, both continuous and discontinuous; secondary critical numbers are found only in discontinuous series. To win a game, or trial, beginning with any number which is not critical, one must draw so as to reduce the number of beads to a critical number at his first draw; thereafter he must draw so as to make the sum of each of his draws and the preceding draw of the opponent, equal to L + H.[7]

[6] Other more difficult series of problems may be obtained by permitting three or more non-consecutive numbers or groups of numbers to be drawn. But the foregoing series furnish enough difficulty for the ordinary graduate student under the conditions of our experiment.

[7] In the discontinuous series this procedure may result in the final re-

B. Apparatus and Recording of Data

The apparatus consisted of a stop-watch, a metronome, an Edison telescribe, or dictaphone, and a row of thirty beads strung upon a steel wire the ends of which were inserted into metal cubes of sufficient height to permit the beads to move freely. These beads were substituted for the matches of the original game because they offered less obstruction to free manipulation, and therefore increased the value of the time records of individual reactions.

While the experiment progressed the dictaphone was in action with the horn so adjusted as to secure as clear a record as possible of all the verbal reactions of the subject. The subject was asked to state aloud the number taken at each draw as the "move" was being made. The experimenter also called out the number taken by him at each draw. Thus the dictaphone records contained all the draws and such comments as were made during the experiment. To mark the time consumed by these reactions a metronome beating half seconds was placed near the dictaphone. The beats were clearly audible in the records. As the experiment progressed the experimenter made a written record of the individual draws of the subject and of the experimenter in separate columns. Incipient movements, indicating that the subject was considering a particular draw, were recorded with a distinctive mark but were not counted in the summing up of data. Upon these written records the dictaphone records were later transcribed. The beats of the metronome were counted, and the number of half seconds from the beginning of the trial to each of its component reactions were recorded. Though no high degree of accuracy is claimed for these time records, they were accurate enough to be exceedingly useful in showing the distribution of attention among the different reactions entering into a trial. They also serve to show differences in the speed of reaction of different subjects and of any one subject in different portions of the experiment. From the dictaphone records were

duction of the number of beads to a basic secondary critical number, which will be reduced to a basic primary critical number by a series of L draws.

also obtained the comments of subjects, which were recorded in their proper places in the records. These verbal reactions were often useful in the interpretation of our data. In order that the learning process might go on as naturally and as free from interruption as possible, no introspections were called for and few were given.

For various reasons it was found necessary in certain portions of the work to dispense with the use of the dictaphone except for the recording of the more important verbal responses. Time measurements were here obtained by means of a stop-watch. In such cases it was of course impossible to record the time of each separate reaction, although the time of complete trials was taken with as great accuracy as was possible by the more cumbersome method. With this procedure it was not possible to distinguish the time consumed by the experimenter's draws from that consumed by the reactions of the subject. To make the records of different subjects comparable in time, therefore, the experimenter made the time of his draws as uniform as possible. A fair estimate of the degree of their uniformity may be made from the record of a relatively small number of typical reactions. In 206 trials with 11 beads taken from the records of various subjects, it was found that the average time per trial was 41.14 metronome beats, or half seconds. The average time per trial consumed by the experimenter's draws was 9.76 half seconds with an average deviation of 1.99. This A.D. of 20 per cent from the average of the experimenter's time per trial is not a serious matter when it is remembered that his time was on the average less than one-fourth of that consumed by the reactions of the subject. The effect of the inclusion of the time required for the experimenter's draws is to put the subject who draws most rapidly at a slight disadvantage, but the time differences in our records are large enough to make these small errors rather unimportant.

C. Order of Presentation of Problems

As already stated, each series consists of what are to the subject at the beginning a succession of relatively independent

problems. Beginning with H + 2 beads in all series in which L is 1, and with H + 2L — 1 beads in all series in which L is greater than 1, the numbers (or problems) of each series were presented in the order of their magnitude from lowest to highest. As soon as a practicable solution was found for one number, the next higher number of the series was presented. Thus successive problems of a series were solved until a satisfactory generalization was formulated for the entire series. A higher series of problems was then presented in the same manner. The order of presentation of series was the order of the magnitude of their L and H draws. First, all series in which L is 1 were presented, beginning with Series 1-2 and proceeding through Series 1-3, 1-4, 1-5, etc., until a general solution for all series of this *order*[8] was found. Series of the next higher order (i.e., in which L is 2) were next presented, beginning with Series 2-3 and continuing through Series 2-4, 2-5, etc., until a general solution for all series in which L is 2, was found. In the same manner successively higher orders of series were presented until a general solution was obtained for all continuous series. The number of continuous series was not the same for all subjects. A workable generalization was made by some subjects after the solution of but a few series in only two or three orders. Others were unable to arrive at a suitable generalization until after the solution of numerous series extending through a number of orders. Only a limited number of discontinuous series was presented, however, and this number was the same for all subjects. These discontinuous series are listed on page 7 in the order in which they were presented.

The order of presentation of the discontinuous series may need a word of explanation. The first portion of this group of series, extending from Series 1 or 3 to Series 4 or 12 inclusive, was given to bring out the fact that secondary critical numbers are found by the addition of multiples of 2L to the primary critical numbers. This is the simplest possible collection of series with a sufficient variety of values for L and H, in which

[8] The *order* of a series refers to the number represented by L. In series of the first order L is 1, in the second order L is 2, etc.

all multiples of 2L above any primary critical number, are critical. Not all subjects succeeded in getting this generalization from this small group of series, but the addition of other series of this character was impracticable owing to the large number of beads that would need to be manipulated. In Series 1 or 4 to 4 or 9 it was expected that subjects would discover that not all multiples of 2L above a primary critical number, are critical; and also find some rule by which to locate such multiples of 2L above primary critical numbers as are not, in a given series, critical.

After the completion of Series 4 or 9 the beads were put aside, and the subject was asked to name the critical numbers in Series 4 or 11 and later in Series 5 or 11. The work in these series was necessarily done mentally and often served to compel the subject to return to his earlier generalizations in the attempt to find a principle which would apply here. If at the end of these series no single workable generalization for all series had been given, the subject was permitted to refer to a table of critical numbers for all the discontinuous series.[9] Here, as in the preceding portions of the work the time required and such of the subject's observations as were given verbal expression, were recorded.

D. Degree of Learning and Uniformity of Procedure

In order to make the conditions uniform for all subjects it was necessary that the experimenter govern his draws in accordance with some definite rules. It was, of course, necessary that no rule should prevent the experimenter from taking advantage immediately of any error made by the subject, but aside from this there was a fair latitude within which he might vary his draws without necessarily affecting the outcome of the game. In Series 1-2, for example, so long as the subject took care to reduce the number of beads to a multiple of 3 at each of his draws, it might seem to be a matter of indifference whether the experimenter took 1 or 2 at a draw. But the indiscriminate changing of his draws would result in a serious lack of uni-

[9] This table with a few changes in the italics is given on page 7.

formity in the condition to be met by the different subjects. If, on the other hand, the experimenter were to take 1 always or 2 always so long as the subject's draws were correct the latter might, by mere mechanical memory and repetition of previous draws, win a sufficient number of consecutive trials to permit him to pass on to the next higher number in the series, and some mechanical performance of this sort might be repeated indefinitely. To prevent such mechanical repetition of accidental successes while still maintaining uniform conditions for all subjects, the experimenter first drew 1 at every draw through an entire trial and then 2 at every draw throughout the next trial. This change in the number drawn by the experimenter in alternate trials was continued until the problem was solved. But if S made an error in any trial, E immediately departed from his uniform reactions and, taking advantage of the error, drew so as to win. The change in E's draw in alternate trials often proved very disconcerting for a time to subjects who tried to repeat accidental successes from memory, but it usually resulted in bringing about some kind of attempt at critical analysis. When two trials had been won in succession by the subject, another bead was added and the process repeated. Thus successive increments of 1 were made and the solutions for the resulting numbers found until a solution for the entire series was obtained.

This changing back and forth of the experimenter's draw in alternate trials, together with the requirement that S win two trials in succession, resulted in bringing out all possible variations in Series 1-2. Two lines of procedure each somewhat in accord with this were possible in the higher series. (1) E might alternate the L draw in one trial with the H draw in the next, neglecting all the intermediate draws except when an error on the part of the subject made it possible for E to win with one of the intermediate draws. (2) He might use *all* of the possible draws, taking each exclusively for an entire trial in some regular order. The latter procedure was used with some of the subjects in the preliminary experiments. It has the advantage of preventing, in a measure, the direction of attention to non-essentials. But, if consistently carried out, it requires that the

subject win each number in the series as many times as there are possible draws in the series and so interferes seriously with the comparability of data from different series. The requirement that S win H minus L consecutive trials with each number also becomes somewhat monotonous in the higher series. The former procedure has the advantage of avoiding this monotony and keeping the degree of learning more nearly uniform throughout all series. It therefore makes the data from different series more truly comparable. This procedure was followed in the work of all of our subjects except a few who served in the preliminary experiments. No attempt is made to compare their work in the later series with that of other subjects.[10]

As soon as a subject recognized two successive critical numbers in a series by inspection, or showed his mastery by prediction of the outcome with higher numbers in the series, or gave a correct generalization, he was presented with a considerably higher number in the same series. He did not draw from this higher number, but merely announced his decision as to what would be the correct draw and gave a reason for his decision. In case the correct draw was indicated but no intelligible reason could be given, the subject was taken back to the point previously attained and required to follow the regular procedure until clear evidence was given of at least some sort of a practical solution which would hold for all numbers in the series. This was not a very common occurrence, however, and never resulted in serious departures from the ordinary procedure. In most cases

[10] Even this procedure cannot, however, be said to make the data from different series entirely comparable. The fact that the critical and the non-critical numbers are not in the same proportion in different series, is bound to affect the degree of learning in some measure. Compare Series 1-2 with Series 1-5 in this respect. In the former series the subject encounters five critical numbers in working from the beginning of the series to and including 15 beads; in the latter he encounters two. In the former series he need not reduce the number of beads to each of these critical numbers more than four times before passing to the next higher critical number. In the latter series he must reduce the beads to each of the critical numbers at least ten times before he is permitted to pass on to the next higher critical number. There is, of course, in the more frequent occurrence of critical numbers in the lower series, some compensation for the higher degree of learning of critical numbers in the higher series.

a verbal formula was given, and sometimes this was expressed in mathematical form.

At the completion of a series the next higher series followed immediately or at the next sitting. The solving of successive series continued until a general solution for an entire order of series was given or until a complete series was solved without a draw. When this occurred, a higher "test" series of the same order was usually but not always given, being omitted only when the subject's statement of his generalization left no doubt that he had found an adequate solution for all series of that order. In case of failure in one of these test series, it need hardly be said, a return was made to the series following the last one which had been solved. If the test series was successfullly solved, the first series of a higher order was presented, and so on throughout the experiment. As already stated, it was impossible to adhere strictly to this uniform degree of learning in case of the discontinuous series. Here the degree of learning differed considerably with different subjects, some of whom persistently worked out the relations existing between successive series whereas others were content to generalize for each series in isolation.

E. Instructions to Subjects

With the apparatus in position and the subject ready to begin work, the experimenter said in substance: "We shall draw alternately from this string of beads. You may take 1 or 2 beads at a draw and I also may take 1 or 2 at any draw. (With each statement of the number to be drawn the experimenter illustrated by manipulation of the beads, showing the alternation between his and the subject's draws which were made from opposite ends of the row of beads.) The object is to get the last draw, i.e., you win if you get the last draw. You always draw first. Will you call out the number you take at each draw?"

The number was then immediately reduced to 4 and the subject asked to begin. The procedure from this point has already been described. As soon as the subject announced his inability to win with 6 beads as the initial number, the experimenter

asked: "Hereafter whenever you make such a discovery or get an idea which seems to be signifcant, will you let me know at once?" The subject was also told at this point, provided his questions had not brought out the fact earlier, that he would be expected to find a general rule for the solution of all numbers in the series. The request that S make known any seemingly relevant new ideas was repeated at the beginning of each of the early series and at other times when his actions seemed to indicate that he had made a new discovery. Subjects were not permitted to record any data though the experimenter's record for a few of the immediately preceding trials was usually in sight. This was unavoidable owing to the difficulties of manipulation and recording. While attempting to recall successful reactions of earlier portions of the work, subjects sometimes asked permission to see their records. This was not permitted. At the beginning of the second series the subject was told: "The requirements of the game are the same as before but you may now take either 1, 2, or 3 beads at a draw." The subjects were of course informed at the beginning of each of the later series as to the numbers which might be drawn. Subjects were asked to refrain from thinking of the experiments during the intervals between their periods of work.

F. The Subjects

Exclusive of those who served in the preliminary experimentation by which the final mode of procedure was determined, 46 subjects served in our experiments. The subjects were divided into groups as follows: (a) Group I consisting of 14 subjects, (b) Group II consisting of 20 subjects, (c) Group III consisting of 12 subjects. The major portion of our study is based upon the work of the subjects in Group I. These subjects will be designated by Roman numerals. Subjects i to x are numbered in the order of the speed of their performance as determined by the number of trials required for the solution of all the problems presented, Subject i being the speediest, Subject ii the next speediest, etc. Subjects xi, xii, xiii, and xiv did not finish the entire experiment and could not, therefore, be given their relative

ranks. Subject iii was a first-year high school boy who was thirteen years old; Subject vi was a college senior, and the remaining ones were all graduate students or instructors in the University of Chicago. All of the subjects of this group were males except Subject v.

Subject ii solved most of the problems at two sittings in one day and returned three weeks later to finish in one short period. Subjects i, iii, iv, vi, vii, and xii were scheduled to work every day, but failed to live completely up to the schedule. Subjects ix and x worked with a fair degree of regularity at weekly intervals, and the remaining subjects of this group were quite irregular in the distribution of their effort. The work with this group of subjects was done at the University of Chicago during the fall and winter of 1916-17.

Group II consists of 20 members of a class of 21 third- and fourth-year college and graduate students at the Kansas State Agricultural College. This group is subdivided into Groups IIa and IIb each consisting of 10 subjects. Group IIa solved first Series 1-2 and then Series 1-3. The members of this group are designated by the capital letters A to J in the order of their speed of learning, the speediest first. Group IIb solved first Series 1-3 and then Series 1-2. The members of this group are designated by the small letters from a to j, again in the order of their speed of learning. The work with Group II was done at the Kansas State Agricultural College in the spring of 1918.

Group III consists of all of the members of a class of third- and fourth-year college students. The subjects in this group worked out solutions for problems in Series 1-2 from 14 to 20 inclusive. The work with this group of subjects was done at the Kansas State Agricultural College during the fall of 1918.

III. RESULTS

A. Unit of Measurement

Before proceeding with the presentation of data it is necessary to attend to the selection of a suitable unit for the measurement of progress. To be entirely comparable successive units of effort should be uniform in (a) time consumed, (b) degree and distribution of attention, and (c) number and character of reactions. Perfect uniformity in all of these characteristics is obviously out of the question, if subjects are to be given the requisite freedom for normal reasoning. There is no constant unit of time into which the work of our subjects can be divided so that the distribution of effort in successive units will conform even approximately to (b) and (c) of the foregoing requirements. Of a number of units of response which suggest themselves the trial[1] conforms most closely to all of these requirements.[2] Measurement and comparison of progress will therefore be made chiefly in terms of the trial, though the number of errors made during the solution of the various problems, as well as the time required, will also be stated and utilized to some extent in the treatment of some portions of our data.

It will be worth while briefly to inquire into the variability of the trial. The characteristics in which variability is greatest and also most susceptible to accurate measurement, are the time and

[1] A series of draws culminating in the reduction of any initial number of beads to 0, constitutes a trial, regardless of who draws last.

[2] It might be supposed that the *draw* would be a better unit of measurement, but this unit is altogether too variable to be of service. In time consumed it varies from less than a half second to several hundred seconds. In the amount of critical attention called forth, it varies from serious, concentrated effort to the most mechanical performance. Often, indeed, the draws do not stand out singly as significant to the subject, but are linked up in various combinations into series each of which constitutes a trial and a real unit in the distribution of attention. Nor can the measurement of progress be made in terms of errors or of successful performances,—i.e., of successes in reducing the number of beads to 0,—since in drawing from a critical number no successes of this objective sort and likewise no erroneous draws, are possible.

number of draws per trial. The facts regarding the variability of trials in these characteristics, as found in the work of our major group of subjects in Series 1-2, are given in Table I.

TABLE I

	No. of Subject's Draws per Trial			Time per Trial		
Subj.	Av. No. Draws	A.D.	V[3]	Med. Time	Q[4]	V[5]
iii	3.13	.95	.30	27	7.0	.26
iv	3.33	.71	.21	39	13.1	.34
i	3.35	.67	.20	41	11.7	.29
v	3.58	.91	.25	58	30.0	.52
ii	3.66	.89	.24	85	34.9	.41
viii	4.20	.68	.16	25	5.4	.22
vii	4.21	1.41	.34	27	10.5	.39
xi	4.25	1.02	.24	54	16.2	.30
ix	4.31	1.29	.29	26	6.8	.26
xiii	5.01	1.68	.33	25	9.5	.38
xii	5.22	1.62	.31	48	26.5	.55
x	6.84	2.89	.42	43	14.6	.33
vi	7.15	1.68	.23	36	10.3	.29
xiv	8.03	3.64	.45	33	14.5	.44

[3] $V = \frac{A.D.}{Av.}$ [4] $Q = \frac{Q^3 - Q^1}{2}$ [5] $V = \frac{Q}{Med.}$

The table is self-explanatory. Note the slight tendency for subjects who rank high in the number of draws per trial to rank low in time per trial. The average of the average numbers of draws per trial for the first seven subjects listed is 3.64 and the average of their median times per trial is 21.5 seconds. The corresponding averages for the last seven subjects listed are 5.83 draws and 16.9 seconds. The correlation between these two lists of averages is —.105.[6] The variations in time and in number of draws per trial do not, therefore, tend to show any cumulative effect upon the relative values of trials in the work of different subjects. Moreover, the distribution of attention is such as largely to discount the value of variations in time and number of draws per trial—especially the latter. It seldom occurred that a subject attended carefully to all of the draws in a trial when the initial number of beads was higher than 17 or 18. One portion of the trial was usually regarded as crucial and the at-

[6] Rank method.

tention mainly directed upon that point with the result that all but a few of the draws in the trial were executed with only a minimum of attention. If these mechanical draws were here neglected, the variation in number of draws per trial would be extremely slight, and the time variations would actually tend to compensate for the variations in the number of draws per trial.[7] Again, after the longer pauses, subjects frequently came back to the work declaring that they had permitted their minds to wander in the pursuit of irrelevant associations. The comments of subjects indicate that such distractions occurred rather frequently. If these distractions are to be regarded as indications of a low degree of attention, we have here another case of an inverse or compensating relationship between variations in different aspects of the trial. Such practice value as the mechanical draws may have possessed was perhaps completely offset by their distracting effect upon attention to points which were regarded as crucial. At any rate this was the spontaneous verdict of a number of the subjects.

The variability of trials within Series 1-2 is typical of that in all later series in which any considerable difficulty was encountered. There was no great variation in the average length and duration of trials from series to series. In the later series—especially those which were presented late in any group of series of the same order—there were generally fewer draws per trial, and attention tended to be more evenly distributed among the various draws of the trial, though mechanical draws practically always appeared in any series where the number of draws per trial ran high. The average time per trial required by Subjects i to x inclusive in some of the more difficult series, is given in Table II. The series reported in this table represent approximately two-thirds of all trials of these subjects in the entire experiment.

[7] This compensating relation in the variations of time and number of draws per trial is not, of course, found in the trials upon higher as compared with those upon lower numbers of a series in the work of the same subject. Here the time per trial usually varies directly with the number of draws though not in the same proportion. But even here it is doubtful whether the longer trials are of more value than the shorter ones, owing to the presence and distracting effect of the more mechanical draws.

TABLE II

Series	No. Trials	Av. Time per Trial[8] (in seconds)	A.D.	V.
1-2	2060	24.09	5.46	.23
2-3	647	42.72	10.68	.25
2 or 6	360	43.15	23.69	.55
1 or 4	423	38.42	14.11	.37
2 or 7	265	42.19	2.94	.07

[8] These averages and deviations are those of all trials in a series without regard to the average time per trial of individual subjects.

There is a general tendency for subjects to exercise a little more caution after the solution of one or two series than at first. This accounts for the marked increase in the average time per trial in Series 2-3 as compared with that of Series 1-2. In the last four series listed the uniformity of the average time per trial is fair, but the deviations are large. However, it should not be overlooked that in the more important matter of degree and distribution of attention the variability, though not susceptible to quantitative statement, is undoubtedly much less marked, and that the variations in one aspect of a trial often tend to compensate for these in other aspects. On the whole, it is safe to say, the trial in the present experiment is not less uniform than the units of measurement regularly employed in studies on maze learning. Its uniformity is probably far greater than that of the trials in certain experiments in ball-tossing, from which curves of learning have been plotted and conclusions of far-reaching consequence drawn.

B. Changes in the Rate and Character of Progress

The progress of the ten subjects of our major group, who solved all of the series of problems, can be traced in Table III. The series of problems are listed at the left in the first column. Following each series are the records of the work of individual subjects upon it. In columns P and E are given the number of problems (i.e. of different numbers within a series) which are solved and the number of errors made in arriving at a general solution of the series. The time is given in seconds.

Table III

Progress of Subjects through all Series of Problems

	Subject i				Subject ii[9]				Subject iii			
Series	P	E	Trials	Time	P	E	Trials	Time	P	E	Trials	Time
1-2	10	24	48	1716	12	8	35	1572	12	5	28	464
1-3	4	0	7	247	9	4	23	585	5	1	10	120
1-4	5	5	14	124	0	0	0	0	0	0	0	0
1-5	0	0	0	19	0	0	0	0	0	0	0	0
2-3	22	11	44	1661	8	3	17	507	11	1	25	427
2-4	0	0	0	10	4	0	4	215	28	13	84	1685
2-5	0	0	0	0	0	0	0	38	16	13	60	1115
2-6	0	0	0	0	0	0	0	0	0	0	0	32
3-4	0	0	0	165	0	0	0	53	0	0	0	15
3-5	0	0	0	123	0	0	0	0	0	0	0	0
4-5	0	0	0	215	0	0	0	50	0	0	0	45
1 or 3	4	—	4	667	8	—	8	269	7	—	11	115
1 or 5	12	—	13	1387	2	—	3	118	6	—	13	173
1 or 7	0	—	0	54	0	—	0	36	4	—	5	62
2 or 6	17	—	21	2483	3	—	3	178	15	—	30	618
2 or 10	0	—	0	483	4	—	4	201	12	—	18	289
3 or 9	7	—	6	381	5	—	6	437	0	—	0	65
4 or 12	0	—	0	120	0	—	0	193	0	—	0	14
1 or 4	4	1	8	492	12	15	38	2701	20	6	47	757
1 or 6	0	0	0	46	24	1	47	4076	14	1	29	467
1 or 8	0	0	0	23	11	1	19	636	0	0	0	193
2 or 5	0	0	0	34	11	0	14	606	6	0	9	361
2 or 7	0	0	0	32	12	0	15	644	16	1	19	835
2 or 8	0	0	0	342					0	0	0	164
2 or 9	0	0	0	78	11	0	15	647	0	0	0	55
3 or 7	0	0	0	35					0	0	0	80
3 or 8	0	0	0	21	11	0	15	664	0	0	0	27
4 or 9	0	0	0	38	0	0	0	310	0	0	0	90
4 or 11				40				365				25
5 or 11				30				230				62
General solution for all series				155				533				70
Total	85	41	165	11215	147	32	266	15864	172	41	388	8425

[9] Series 2 or 8 and 3 or 7 were accidentally left out of the experiment with this subject.

TABLE III (Continued)

	Subject iv				Subject v				Subject vi			
Series	P	E	Trials	Time	P	E	Trials	Time	P	E	Trials	Time
1-2	11	55	96	2205	13	23	62	2834	27	221	331	7934
1-3	12	13	35	704	8	1	15	363	11	6	30	265
1-4	10	6	27	449	2	0	3	33	2	0	4	54
1-5	6	0	10	87	0	0	0	9	0	0	0	40
1-6	14	0	24	241	0	0	0	0	0	0	0	0
1-7	8	0	14	157	0	0	0	0	0	0	0	0
1-8	0	0	0	36	0	0	0	0	0	0	0	0
2-3	13	16	44	1800	21	19	71	4566	15	5	30	709
2-4	6	1	13	279	13	7	29	894	1	0	1	268
2-5	0	0	0	30	0	0	0	54	0	0	0	0
3-4	0	0	0	25	0	0	0	17	1	0	0	205
3-5	0	0	0	0	0	0	0	28	0	0	0	74
4-5	0	0	0	26	0	0	0	10	0	0	0	75
1 or 3	7	—	17	180	4	—	4	52	10	—	10	237
1 or 5	5	—	9	110	10	—	18	377	14	—	18	503
1 or 7	2	—	3	33	10	—	14	269	13	—	14	235
2 or 6	27	—	45	1818	33	—	46	1189	24	—	36	821
2 or 10	6	—	11	546	25	—	32	916	11	—	14	262
3 or 9	4	—	6	151	0	—	0	72	13	—	13	258
4 or 12	0	—	0	75	0	—	0	165	9	—	8	142
1 or 4	12	7	26	430	13	3	21	392	19	1	31	484
1 or 6	13	5	24	1006	23	5	47	1657	6	0	8	147
1 or 8	0	0	0	58	16	5	28	984	2	0	2	66
2 or 5	0	0	0	152	8	0	8	427	16	2	21	346
2 or 7	8	0	14	590	21	2	34	1631	6	0	6	145
2 or 8	2	0	2	333	7	0	10	527	3	0	4	75
2 or 9	0	0	0	58	10	0	15	993	0	0	0	66
3 or 7	17	6	33	1526	0	0	0	285	8	0	10	159
3 or 8	0	0	0	600	0	0	0	192	3	0	2	58
4 or 9	0	0	0	604	0	0	0	40	0	0	0	165
4 or 11				820				85				65
5 or 11				55				290				35
General solution for all series				8610				1210*				280
Total	183	109	453	23794	227	65	457	20561	214	235	594	14173

TABLE III (Continued)

	Subject vii				Subject viii				Subject ix			
Series	P	E	Trials	Time	P	E	Trials	Time	P	E	Trials	Time
1-2	19	182	371	6659	15	90	168	2591	28	187	315	6129
1-3	12	3	23	754	12	24	56	684	6	4	16	244
1-4	5	0	8	207	10	8	30	294	2	0	3	23
1-5	0	0	0	15	1	0	2	23	1	0	2	52
1-6	0	0	0	0	0	0	0	15	4	0	9	115
1-7	0	0	0	0	0	0	0	0	0	0	0	6
2-3	18	1	25	1827	69	27	167	5408	42	25	119	5859
2-4	9	0	13	1047	35	9	71	2679	25	4	53	1956
2-5	0	0	0	28	15	0	20	500	10	1	15	295
2-6	0	0	0	0	2	0	2	59	0	0	0	35
2-7	0	0	0	0	0	0	0	9	0	0	0	0
3-4	8	0	12	301	19	4	41	699	9	0	10	155
3-5	0	0	0	63	3	0	4	75	6	0	9	118
3-6	19	1	30	3115	0	0	0	13	0	0	0	20
3-7	0	0	0	21	0	0	0	0	0	0	0	0
4-5	0	0	0	12	0	0	0	21	0	0	0	0
1 or 3	6	—	12	372	24	—	33	1067	17	—	36	1101
1 or 5	8	—	11	327	19	—	24	870	5	—	11	195
1 or 7	0	—	0	6	3	—	4	76	1	—	2	52
2 or 6	19	—	19	2427	23	—	38	1268	51	—	108	4189
2 or 10	7	—	16	979	29	—	34	755	17	—	20	489
3 or 9	0	—	0	60	19	—	20	309	14	—	18	680
4 or 12	0	—	0	80	0	—	0	52	0	—	0	87
1 or 4	14	1	22	1583	24	2	39	877	37	41	154	7230
1 or 6	12	1	19	4691	12	0	16	303	14	0	24	1318
1 or 8	0	0	0	30	0	0	0	40	14	0	21	947
2 or 5	11	2	20	728	24	8	46	1441	11	1	19	758
2 or 7	7	0	12	614	48	12	86	3702	12	3	27	966
2 or 8	7	0	13	954	9	2	14	263	9	2	13	446
2 or 9	8	0	10	1102	12	1	15	271	10	0	15	660
3 or 7	7	2	13	293	26	2	34	677	10	0	21	360
3 or 8	8	0	8	678	11	0	14	369	10	0	17	369
4 or 9	4	0	5	828	8	0	9	278	11	1	22	589
4 or 11				455				690				240
5 or 11				1105				910				150
General solution for all series				4980				2420				7740*
Total	208	193	662	36341	472	189	987	29708	376	269	1970	43753

TABLE III (Continued)

Subject x

Series	P	E	Trials	Time	Series	P	E	Trials	Time
1-2	47	313	597	17538	2 or 10	7	—	7	233
1-3	16	3	45	1122	3 or 9	27	—	28	980
1-4	2	0	0	136	4 or 12	9	—	8	514
1-5	0	0	0	0	1 or 4	19	11	37	1306
2-3	36	16	105	4881	1 or 6	22	14	49	1421
2-4	15	0	26	728	1 or 8	26	5	36	1973
2-5	3	0	2	77	2 or 5	14	3	21	1016
2-6	0	0	0	0	2 or 7	35	4	50	1922
3-4	37	0	40	240	2 or 8	9	0	12	377
3-5	1	0	0	230	2 or 9	1	0	0	118
3-6	10	0	12	444	3 or 7	19	1	21	968
317	0	0	0	28	3 or 8	12	1	14	548
4-5	11	0	10	570	4 or 9	12	2	18	485
5-6	0	0	0	50	4 or 11[11]				327
1 or 3	21	—[10]	23	858	5 or 11				193
1 or 5	8	—	10	466	General solution				
1 or 7	3	—	4	225	for all series				1960*
2 or 6	13	—	14	545	Totals	435	373	1192	42470

The most striking feature of these figures is the great irregularity in the number of trials, the time required, etc., for the solution of various problems. Note the relatively large number of trials required in Series 1-2, and the rapid decrease in the number of trials in successive series of the first order.[12] The same sort of change is found in series of the second order and, to some extent, in all higher orders of the continuous series as well as in those of the discontinuous series. These regular and more or less gradual changes are due in part to the separate mastery of numerous relatively isolated though often poorly defined "elements" of the problematic situation; in part they are due to a gradual definition and development of larger and more complex units in the form of concepts and general principles.[13]

[10] No errors are possible in those series where blanks appear in the error column.

[11] Subjects were required to work Series 4 or 11 and 5 or 11 "mentally."

* These subjects failed to find a general solution for all series in the time at their disposal.

[12] The *order* to which a series belongs is determined by the value of L. In series of the first order L is 1; in those of the second order L is 2, etc.

[13] Any aspect or isolable portion of the objective situation and any relationship between such portions or aspects is here regarded as an element of the problem.

These gradual changes are sometimes obscured by irregular and often extremely abrupt changes arising from various causes, such as, distraction of attention, sudden utilization of old concepts, monotony, fatigue, rest, elimination of erroneous assumptions, change of methods, etc.

The more striking irregularities in the rate of progress are shown in graphic form in Figure I. The number of trials and the amount of time required as well as the number of errors made by each of the ten subjects in various series and groups of series, is represented by the columns in the figure. The series and groups of series represented in the figure are indicated at the left. The subjects are designated by the numerals at the base of columns.

This irregularity is in some degree illusory owing to the failure of some bonds in the early stages of their development to register their effect in terms of those particular overt responses which were taken into account in the score. That bonds do begin to develop long before their effect is apparent in the crude score, is evident from the comment of subjects, the comparative length of delays, incipient reactions, etc., some of which are included in the detailed records and will be noted later. Further evidence of the inadequacy of any one sort of measure to register all of the changes in the strength of bonds, is found in the relation between the time, the errors, and the number of trials involved in the solution of successive series of problems. In Figure I the number of errors made and the amount of time consumed in the solution of the problems in each section of the figure are shown by the relative height of the first and third columns respectively. Observe that in every case in Series 1-2 the error column is higher than the time column, and that there is a somewhat gradual shifting in the relative height of these columns so that in the last group of series the time column is in every case higher than the error column.[14] This change is to

[14] The relation between the trial column and the time and error columns in Series 1-2 and in the last group of series, is worthy of note. In every case but one the height of the trial column is intermediate between that of the other two columns. The measurement of progress in terms of trials therefore takes special account of the conflicting claims of time and errors where the conflict is greatest.

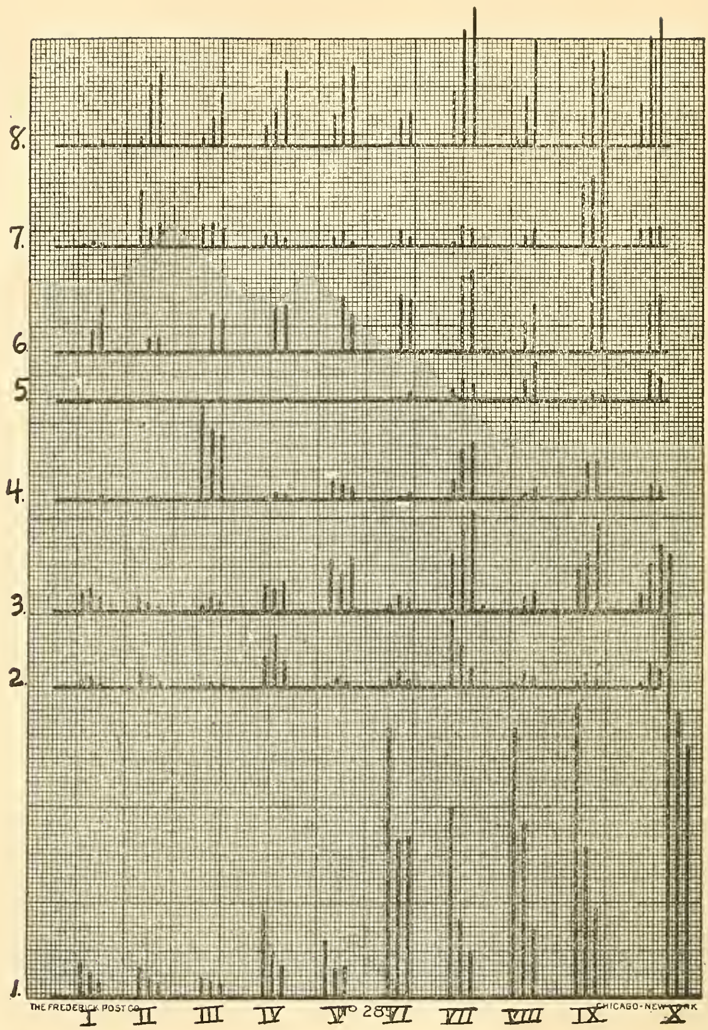

FIGURE I.—The Roman numerals at the base of the columns of the lowest section represent the various subjects of Group I, who solved all series. The height of the middle column for each subject represents the number of trials required by him for the solution of the problems indicated by the number at the left. The *trial columns* are all directly comparable. The number of errors made and the amount of time required by each subject are represented by his first and third columns respectively. The time and error columns are drawn to such a scale for each subject as to make their total length for all series equal to the total length of the trial columns of that

some extent influenced by increased caution, as is shown by the lengthened average time per trial in the later series as compared with the first series; but it is due mainly to the fact that performance represents a distinctly lower plane of learning here than does formulation. It is a little surprising to find this relation between power of performance and power of formulation here in view of the simplicity and definiteness of the terms which are required for formulation. It is possible that if success were to demand a high speed of reaction, as is generally the case in acts of skill and to a large extent in practical thinking, this relationship would tend to disappear. However, with the time allowed for each reaction limited only by his patience, the subject is usually able to respond correctly while he is yet unable to foretell the correct response in the absence of the appropriate concrete situation or to say why the response is correct when made. This disparity in power of performance and power of formulation as well as some of the more striking cases of change in speed of progress, both of the regular and of the irregular sort, will receive more special attention in later sections.

The principal features of the progress from series to series are duplicated in the progress from problem to problem in some of the more novel series. There are, however, some important differances. Series 1-2 being the first and most novel series illustrates the natural progress from problem to problem better than do any of the later ones. The progress of the fourteen subjects of the major group through the problems of this series,

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particular subject. They are thus directly comparable with one another and with the trial columns of the same subject. But the error and time columns of *different* subjects are *not* directly comparable.

The numbers at the left represent various series and groups of series of problems as follows:

1. Series 1-2.
2. All later continuous series in which L is 1.
3. Series 2-3.
4. All later continuous series in which L is 2.
5. All continuous series in which L is 3 or greater.
6. All discontinuous series from Series 1 or 3 to Series 4 or 12 inclusive
7. Series 1 or 4.
8. All remaining discontinuous series.

is shown in Table IV. The problems of the series are listed at the left in the column headed "Initial No. of Beads." The number of trials and of seconds required by each subject for the solution of each problem are listed in self-explanatory form.

The errors are not included in this table but their number is roughly proportional to the number of trials per problem except in the critical numbers, where no errors are possible. Certain points of difficulty for various subjects are revealed by the large number of trials and the amount of time required for the solution of some of the problems. Such points of unusual difficulty are found in the record of Subject iv at 10 beads, in that of Subject i at 11 beads, and in that of Subject vi at 25 beads, etc. Sometimes the troublesome problems are so distributed as to suggest delnite, regularly recurring types of difficulty. This is illustrated in the record of Subject vi where numbers which are equal to 1 plus any multiple of 3 offer greater difficulty than those which are equal to 2 plus a multiple of 3. Problems of the former type will be referred to as L-problems and those of the latter type as H-problems. It will be noticed that the L-problems from 7 to 22 inclusive required 72 trials for their solution whereas the H-problems from 8 to 23 inclusive required only 25 trials. The solution of 25 beads which is an L-problem, required but 6 trials. Only in two instances did this subject solve an L-problem with

Table IV.—Progress of Subjects through Series 1-2

Initial No.	Subj. i		Subj. ii		Subj. iii		Subj. iv		Subj. v	
of Beads	Trials	Time	Trials	Time	Trials	Time	Trials	Time	Trials	Time
4	2	21	2	29	3	42	4	32	2	20
5	2	13	2	29	2	24	4	28	2	17
6	4	48	2	35	3	36	3	44	2	32
7	2	25	2	97	3	27	7	117	6	89
8	4	59	3	96	4	56	4	79	4	78
9	5	86	3	70	1	21	9	148	11	240
10	5	104	2	79	3	50	50	1151	7	256
11	18	719	6	375	3	81	4	152	11	577
12	4	102	3	112	2	42	0	15	2	117
13	4	539	2	143	2	32	8	270	3	106
14	0	0	7	355	2	29	3	169	7	647
15	0	0	1	152	0	24	0	0	2	252
16	0	0	0	0	0	0	0	0	3	406
Total	48	1716	35	1572	28	464	96	2205	62	2834

Table IV (Continued)

Initial No. of Beads	Subj. x Trials	Subj. x Time	Subj. ix Trials	Subj. ix Time	Subj. vi Trials	Subj. vi Time	Subj. vii Trials	Subj. vii Time	Subj. viii Trials	Subj. viii Time
4	4	35	4	22	2	4	4	40	2	15
5	4	42	2	14	2	4	4	68	2	21
6	4	94	9	85	8	33	22	190	7	60
7	4	62	3	25	11	87	74	533	2	16
8	4	77	3	25	5	46	4	35	4	36
9	17	292	4	35	7	56	12	343	8	88
10	8	108	4	44	5	45	48	664	15	158
11	24	339	68	822	6	68	11	145	32	395
12	28	423	15	192	8	86	26	335	18	193
13	100	2067	6	94	15	211	12	156	33	559
14	9	204	76	1455	2	30	18	257	24	551
15	44	1229	3	54	7	101	36	590	12	264
16	48	1165	88	1799	10	151	11	237	3	104
17	48	1220	3	126	7	108	21	428	3	91
18	15	369	3	76	5	84	16	567	3	40
19	21	687	3	135	26	468	7	202	0	0
20	21	446	2	72	3	42	24	538	—	—
21	3	58	1	47	10	154	2	265	167	2591
22	38	1054	4	248	5	76	9	1064		
23	3	67	2	82	2	28	—	—		
24	6	154	2	34	7	123	371	6659		
25	5	121	2	137	156	5061				
26	3	78	2	84	6	126				
27	8	264	1	48	10	450				
28	2	68	2	114	4	100				
29	9	262	2	152	2	32				
30	5	141	0	19	0	157				
31	10	337	1	87	0	0				
32	6	244	0	0	—	—				
33	10	325	—	—	331	7934				
34	5	181	315	6129						
35	2	95								
36	5	187								
37	17	696								
38	2	77								
39	3	151								
40	7	284								

Subj. x (Continued)

No. of Beads	Trials	Time
41	2	100
42	2	114
43	7	480
44	3	237
45	0	1110
46	18	1212
47	2	151
48	0	19
49	6	467
50	4	400
	—	—
	597	17538

Table IV (Continued)

Initial No. of Beads	Subj. xiv Trials	Subj. xiv Time	Subj. xii Trials	Subj. xii Time	Subj. xiii Trials	Subj. xiii Time	Subj. xi Trials	Subj. xi Time
4	4	18	2	17	2	16	2	9
5	4	42	2	9	3	13	3	40
6	3	59	5	58	5	49	1	31
7	3	44	6	69	4	26	2	25
8	5	86	7	137	3	22	2	51
9	11	124	8	99	6	49	1	33
10	24	352	17	295	8	63	7	182
11	36	568	4	116	30	338	8	225
12	12	123	9	156	38	481	2	53
13	4	52	11	250	2	31	13	419
14	2	14	28	1016	4	89	2	53
15	10	131	7	256	6	103	1	8
16	5	93	39	1622	10	183	2	56
17	14	230	5	393	3	38	2	57
18	10	150	0	127	5	125	0	41
19	4	16	5	253	2	38	3	83
20	9	254	4	211	7	217	2	55
21	12	218	0	63	2	96	0	60
22	5	78	4	170	5	132	0	0
23	7	152	2	97	2	105	—	—
24	3	51	0	2	2	108	53	1481
25	13	295	3	114	4	132		
26	2	38	2	65	2	70		
27	4	68	0	1	2	216		
28	7	146	2	152	3	87		
29	2	37	2	58	2	76		
30	3	62	0	88	1	27		
31	14	355	2	91	3	136		
32	6	214	2	74	0	0		
33	2	68	0	22	—	—		
34	25	758	2	49	166	3066		
35	2	74	2	39				
36	2	72	0	5				
37	23	793	2	106				
38	5	117	0	0				
39	0	1	—	—				
40	7	373	184	6283				
41	3	134						
42	1	97						
43	18	678						
44	2	60						
45	0	4						
46	4	149						
47	4	148						
48	0	6						
49	4	116						

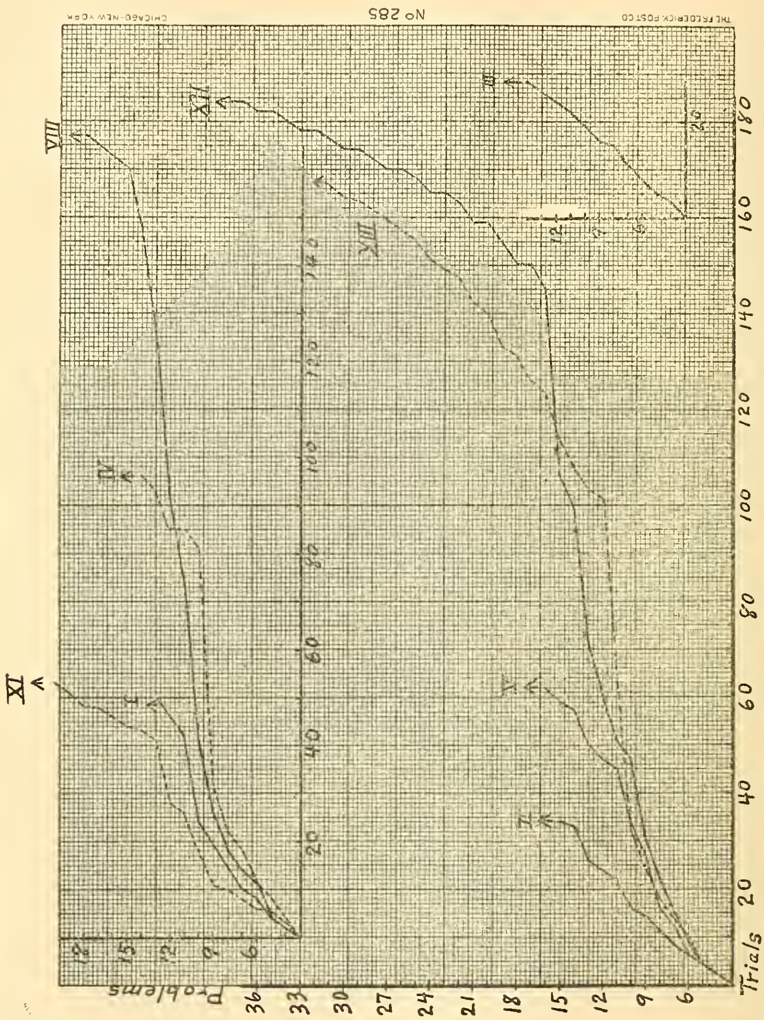
Subj. xiv (Continued)

No. Beads	Trials	Time
50	3	81
4	2	11
5	2	9
6	0	2
7	5	84
8	4	47
9	1	16
10	3	23
11	3	58
12	0	3
13	2	14
14	2	17
15	0	1
16	2	9
17	2	24
	—	—
	367	8171

fewer trials than were required for the preceding H-problem. A total of 236 trials was required for the solution of 9 L-problems and only 35 for 9 H-problems. Subject x required 296 trials for the solution of the L-problems of this series and only 146 for an equal number of H-problems. Subject xiv experienced a change in the type of difficulty at about the middle of the series. The L-problems from 7 to 22 inclusive required only 45 trials whereas the H-problems from 8 to 23 inclusive required 73 trials. But from this point to 44 beads the L-problems became the more difficult, requiring 107 trials as compared with the 22 trials required for the solution of an equal number of H-problems in this portion of the series. A similar change in the type of difficulty occurs in the record of subject vii. The L-problems below 12 required 126 trials while the H-problems were solved in 19 trials. Yet the numbers of trials required for the solution of an equal number of L- and H-problems above this point were 30 and 83 respectively.

The principal causes of these irregularities are found to be false assumptions; erroneous generalizations, founded usually upon very incomplete analysis; uncritical applications of methods and generalizations from preceding problems, and, in some of the extreme cases, monotony and fatigue. Some of these irregularities will be considered at greater length in later sections on generalization and transfer.

Another noteworthy feature of the progress of most of the subjects through the problems of Series 1-2 is the low number of trials upon the early and the late problems of the series as compared with the number of trials required for the solution of some of the intermediate problems. The extent to which this feature characterizes the work of various subjects is shown in Figure II. The number of trials is represented on the abscissae and the number of problems solved—or recognized as insoluble in case of the critical numbers—is represented on the ordinates. Note that with the exception of the curve of Subject iii, all curves show a period of relatively rapid advancement followed by one of slower progress, after which a final period of rapid progress terminates with a successful generalization for the entire series.



No 285
Problems
36
33
30
27
24
21
18
15
12
9
6
Trials
20
40
60
80
100
120
140
160
180

FIGURE II.

The three stages in the learning process represented by these features of the curves are also clearly indicated by the behavior of the subjects and by their comments. The period of rapid initial rise of the curves was obviously a period of perceptual solutions. During the intermediate period of slower advancement the abstraction of significant elements of the problematic situation occupied the greater portion of the energies of the subjects. Some combination among these elements of course occurred immediately, but it was not until after considerable experience with the separate elements that most subjects were able to combine and organize them into effective means of control, capable of altering markedly the direction of the curve and leading speedily to a satisfactory generalization for the entire series. The types of mental process which in some degrees dominated these various stages of learning may profitably be treated separately in the three following sections.

C. Perceptual Solutions

At the beginning of Series 1-2 comparatively few erroneous draws were made, but the percentage of errors increased rapidly for some time with the increase in the number of beads. This increase in the percentage of errors is shown in Table V. This table includes all of the reactions of thirteen subjects upon the non-critical numbers from 4 to 11 inclusive.

Table V

Number of Beads	4	5	7	8	10	11
Total Draws from each Number	222	194	282	262	297	401
Erroneous Draws from each Number	14	46	79	119	92	173
Percentage of Errors	6.4	23.7	27.8	45.4	30.9	43.1

Note the rapid increase in the percentage of errors as the number of beads advances from 4 to 8 and the high but fluctuating percentage beyond 8. The unevenness of this increase is also worthy of note. From 5 beads the percentage of erroneous draws is 17.3 per cent higher than from 4; from 8 beads it is 17.6 per cent higher than from 7. But the percentage of erroneous draws from 7 is only 4.1 per cent higher than from 5 though the interval between 5 and 7 is twice as wide as that between the members

of the other pairs of numbers compared. This difference is undoubtedly due to the nature of the errors which are possible in drawing from the different numbers. The only errors possible in drawing from 4 and from 5 beads result in leaving 2 and 4 beads respectively; from 7 and 8 the only possible errors result in leaving 5 and 7 beads respectively. Thus from the point of view of the number of beads left after an erroneous draw from each of the foregoing numbers, the long intervals occur where the change in the percentage of errors is greatest. The change in the percentage of errors in the early numbers of the series is therefore apparently due to the degree of ease with which the consequences of a given move may be foreseen in a purely perceptual manner.[15]

The perceptual character of these early solutions is further shown by the behavior of the subjects. Usually 1 or 2 beads were moved aside tentatively or otherwise marked off so that the possible result of further draws could be directly perceived.[16] This tentative manipulation was purely a trial and error affair at first, as was indicated by the constant shifting from one of the possible draws to another, as well as by the comments of those subjects who mentioned the matter at all. Sometimes both hands were employed in this sort of manipulation,—one representing the subject and the other the experimenter,—and so the consequences of all possible draws were figured out on the perceptual level. The movements used to mark off, or temporarily exclude some of the beads from attention, were not always of this overt character. Often they became almost imperceptible and occasionally verbal reactions served this function even in the early stages of the game. For some subjects these perceptual judgments were important factors in the solution of problems far beyond the immediate span of attention. Thus some of the

[15] The large downward fluctuation in the percentage of errors at 10 offers no serious difficulty to this view. This fluctuation occurs where it might most reasonably be expected; i.e., immediately after one of the long intervals and after the range of direct perceptual control has been passed.

[16] If the hold upon the beads thus tentatively drawn aside was released the maneuver was counted a draw; therefore the caution, and also the difficulty of applying perceptual checks upon the higher numbers.

subjects, upon discovering late in the game that 6 or 9 could not be won, would mark off and exclude so many beads from attention and then proceed to solve the remaining numbers upon the perceptual level. But usually when the number of beads was increased to beyond 7 or 8, the perceptual form of solution more or less completely broke down.

The important feature of these perceptual solutions is the fact that all progress was here made through trial and error perceptually checked and entirely without the use of symbols except an occasional word used in a very specific way. That is to say, there was a conspicuous absence of generalization in these early perceptual solutions, due to the lack of any necessity for the use of symbols. As will be seen later, the failure to utilize symbols in these early problems seriously limited the transference of control from the lower to the higher problems of the series.

D. Analysis

Immediately following upon the period of perceptual solutions there was usually a period of evident confusion. Though subjects had been instructed at the beginning of the series to look for underlying principles, they generally failed up to this point of the game to see any relation between problems and to remember how or understand why certain numbers had been won. In accordance with James's view that all analysis depends upon the "law of varying concomitants" or upon the elements having somehow previously been brought to attention in isolation, further progress here would require either that the solutions of earlier numbers be recalled and applied to the problems at hand or that some sort of manipulation be carried on by means of which the significant elements of the various problems could be abstracted and associated with appropriate reactions. Both of these alternatives were tried, the former without success in a single case when attempted early in Series 1-2, the latter with varying degrees of success, depending on the individual who made the attempt and the sort of manipulation resorted to.

The sorts of analysis which occurred are classified from the points of view of (1) the specific elements and types of elements

which were abstracted and employed for generalization, (2) the explicitness and extent of analysis, and (3) the temporal relations of manipulation and ideational analysis.[17]

1. *Types of Elements Abstracted: The Direction of Analysis.*

Any aspect or isolable portion of the objective situation and any relation between such portions or aspects will hereafter be referred to as an "element" of the general problem. Elements of this sort appeared in considerable variety and exhibited uniformities of such varied types that different subjects might conceivably have arrived at equally valid solutions from quite different lines of approach.

Subjects usually began early in Series 1-2 to count the number of beads from which they were required to draw at each move. This usually led to an early discovery of the fact that certain numbers are especially significant as points of orientation and control in the series. Certain multiples of 3 were usually the first numbers to take on this special significance in Series 1-2. Thus subjects almost invariably came to regard the numbers which they could not win as the important elements to be sought out. Sometimes subjects also tried to remember the numbers which they were able to win, but these numbers were seldom made the objects of special attention and the basis of hypotheses and generalizations, except as they were brought in to complete the formulation after the solution had been practically worked out upon some other basis.

Perhaps the next most common type of element to attract special attention was the relationship of the draws made by the experimenter to those of the subject. Occasionally a subject would become so absorbed in the pursuit of this relation that he would utterly fail for a time to notice the significance of the number of beads presented. Most of our subjects, however, noticed some sort of relationship here rather early and divided their attention between it and the number element mentioned above.

Other elements of particular interest at times to various sub-

[17] See Ruger, "The Psychology of Efficiency," pp. 10-14.

jects were the number of beads obtained by the subject or by the experimenter or both, or sometimes the total number of beads drawn, or the number of draws obtained by the experimenter or by the subject, or the relation between the number of draws and the number of beads obtained, etc. Though elements of this sort all exhibit uniformities of such character as might well become the basis for successful generalizations, no one succeeded in getting a successful solution from these elements alone, although some subjects lost a considerable amount of time in the attempt.

Without going further into detail it may be said that the more successful subjects usually began very early to give special attention to the number of beads remaining after each draw and often failed entirely to notice the relation between the draws of the subject and those of the experimenter. Often in the first series, and quite generally in the later series, elements of both of these types were constantly taken into account with good results. The attention of the less successful subjects usually fluctuated considerably between the different types of elements, but failed to follow up any type consistently enough to discover the uniformities lying beneath the surface.

Table VI shows the point in the first series where each subject gave the first evidence of having become definitely aware of the uniformities in the elements of each of the first two types mentioned above. This, of course, implies a considerable amount of previous attention to the elements underlying the uniformities. For example a subject would often study his own successful draws and those of the experimenter for some time before becoming aware of the "opposite"[18] relation between them; or he might realize for a considerable time that certain definite numbers are critical without noticing that they are multiples of 3.

It will be noticed that a number of the most successful subjects seemingly failed to discover the principle of drawing by

[18] In all trials beginning with a critical number E drew 1 when S drew 2, and 2 when S drew 1. Likewise, in order to win any non-critical number, S had to draw the "opposite" of E's preceding draw throughout the trial after first having reduced the number of beads to a multiple of 3 at his first draw.

TABLE VI

Subject	Point in Series 1-2 where the principle of drawing by "opposites" was discovered		Point in Series 1-2 where it was discovered that the C numbers are multiples of 3	
i	——	————[20]	13b[19]	50th trial
ii	——	————	15b	35 " "
iii	13b	25th trial	15b	28 " "
iv	9b	25 " "	15b	96 " "
v	——	————	16b	63rd "
vi	14b	71st trial	28b	384th "
vii	6b	21 " "	18b	310 " "
viii	18b	168 " "	17b	162nd "
ix	23b	303rd "	31b	315th "
x	12b	77th "	40b	547 " "
xi	——	————	21b	53rd "
xii	13b	70th trial	10b	30th "
xiii	12b	86 " "	15b	111 " "
xiv	11b	88 " "	40b	293rd "

[19] 13b, 15b, etc., stand for the various problems in the series and indicate the initial number of beads presented at each trial in the problem.

[20] The principle of drawing opposites was not discovered in this series by Subjects i, ii, v, and xi.

opposites, and that some of the subjects who found the greatest difficulty in solving the problems discovered this principle very early in the series. By the rank method the correlation between the number of trials required for the discovery of this principle and the number of trials required to find a satisfactory solution for the series, is —.324. There is a positive correlation of .728 between the number of trials required to discover that the critical numbers are multiples of 3 and the number of trials required to find a satisfactory solution for the series. Attention on the part of the subject to the numbers of beads from which he must draw thus leads more directly to fruitful generalizations than does attention to the relation of his own *draws* to those of the experimenter.

What factors, it may be asked, determine which of the various elements shall be abstracted from the total situation in any case? This question cannot be answered fully from the data at hand but it seems worth while to point out that the order of abstrac-

tion of elements is exactly what might be expected if the principal selective factors were the relative frequency of occurrence of various situation and response elements, the relative nearness of the various elements to a goal, or end of action, and the speed of the subject's reactions. These three factors will be discussed separately in a later section. Some data concerning them may well be given here.

a. Frequency of Repetition.—As already noted the numbers in Series 1-2 which were first to attract special attention and to become objects of active research were almost invariably the critical numbers. This emphasis upon the critical numbers was the result not of sudden, comprehensive insight, but of a gradual and often tedious growth of meaning. There were, to be sure, sudden spurts and long plateaus in the progress of some subjects, as a result of the drawing in of old concepts by association. But the regularity of progress in the abstraction of elements in the first series was not often seriously obscured by these factors. The first number to be isolated and treated as especially significant was almost invariably 6.[21] Thereafter 9, 12, and the higher critical numbers followed in the order of their magnitude except as the order was affected by the influx of old concepts in the form of generalizations. The early stages of the abstraction of the various critical numbers progressed with surprising independence. The first objective signs of the process of abstraction appeared usually in the form of short delays, exclamations, and other indications of critical reaction, which, if we may trust the comments of subjects, were accompanied by occasional fleeting insights into the nature of the situation. Continued repetition of these numbers served to give further emphasis to them and greater depth and stability to the erstwhile fleeting insights into their signifiance for the solution of other numbers, until finally it became possible to formulate their relations into satisfactory principles of control.

[21] It is true that prior to the recognition of the critical significance of 6, 3 was usually recognized in the concrete as a losing number. But, owing apparently to the easy perceptual control of 3, it was almost never mentioned explicitly as a losing number, except as an afterthought in rounding out a generalization which had been made upon the basis of higher critical numbers.

Corresponding to the gradual abstraction of certain critical numbers and the order in which these numbers were affected by the process, is the gradually increasing excess of the subjects' reactions to critical numbers over their reactions to non-critical ones. Not only does this excess of draws from critical numbers increase from problem to problem, but the ratio of draws from critical to those from non-critical numbers also increases rather steadily. The regularity of this increase, both in the excess of draws from critical numbers and in their ratio to the draws from non-critical numbers, may be seen in Table VII. At the left are listed the various problems of the series from the first problem, in which 4 beads are presented, to the twelfth problem, in which 15 beads are presented for solution. The figures in the succeeding columns indicate the total number of times ten subjects were required to draw from each of the numbers listed above the columns, in the solution of the problem opposite which the figures occur and all preceding problems. Thus in the solution of the problem, 4b, the subjects drew 12, 14, 0, and 28 times from 1, 2, 3, and 4 beads respectively. In the solution of 4b and 5b they drew 23, 26, 5, 28, and 26 times from 1, 2, 3, 4, and 5 beads respectively, and so on. Or reading *down* the column, under 3 for example, the ten subjects drew 0 times from 3 while working upon 4b; 5 times while working upon 4b and 5b; 46 times while working upon 4b, 5b, and 6b, and so on. The ratios of draws from critical numbers to the averages of those from adjoining non-critical numbers, are given in italics.

The frequency of subjects' draws from any critical number does not greatly exceed that of their draws from adjacent non-critical numbers until after the game has progressed to a point considerably beyond the critical number in question. Thus the frequency of draws from 3 beads becomes greater than from 2 or 4 beads only after considerable work upon 6 beads; and the frequency of draws from 6 beads becomes greater than from 5 or 7 beads only after some work upon 9 beads, etc. This is in entire accord with the fact that the status of these numbers (i.e., whether critical or non-critical), was in most cases forgotten shortly after the commencement of work upon higher numbers,

TABLE VIII

Total Number of Draws of ten Subjects from various Numbers in Solving the first twelve Problems of Series 1-2.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4b	12	14	0	28											
5b	23	26	5 *.19*	28	26										
6b	23	26	46 *1.70*	28	26	41 *1.58*									
7b	35	41	61 *1.47*	42	43	41 *.94*	44								
8b	44	54	77 *1.45*	52	58	55 *1.08*	44	38							
9b	44	54	152 *2.87*	52	58	130 *2.55*	44	38	75 *1.97*						
10b	65	69	241 *3.28*	78	79	195 *2.32*	89	88	75 *.69*	131					
11b	108	78	408 *4.08*	122	101	333 *2.66*	149	151	172 *1.22*	131	220				
12b	108	78	533 *5.33*	122	101	458 *3.66*	149	151	297 *2.11*	131	220	125 *.56*			
13b	137	98	669 *5.35*	152	124	578 *3.66*	192	186	385 *2.03*	194	291	125 *.52*	186		
14b	163	109	758 *5.28*	178	137	657 *3.62*	226	208	447 *2.01*	236	339	161 *.61*	186	126	
15b	163	109	845 *5.89*	178	137	744 *4.09*	226	208	534 *2.41*	236	339	248 *.96*	186	126	87 *.69*

(See preceding paragraph for explanation of table.)

and was re-discovered only after a considerable number of trials during which the frequency of repetition of forgotten critical numbers became gradually more preponderant over that of forgotten non-critical numbers. Thus, after working for a while upon 7 or 8 beads, the subject usually forgot that he had been unable to win 6 and did not often re-discover the fact until after some work upon 9 beads or even upon higher numbers; that is, until the number of draws from 6 had come to be greatly in excess of those from 5 or 7 beads. The behavior of subjects toward 9 and 12 beads, and often toward higher numbers, was

similar in this respect. But, owing to the increasing influence of generalization and other less conscious sorts of transfer, the preponderance of draws from the critical numbers became gradually less marked or entirely vanished.

Though in the work of these subjects the order in which the various critical numbers acquired their special significance is closely paralleled by the relative frequency of their presentation, it might be questioned whether this order was not determined by the order of their original presentation as problems of the series rather than by the frequency of subjects' reactions to them. That the order of the original presentation of these numbers as separate problems is not the determining factor, however, is evident from the facts presented in Table VIII where the critical numbers were not presented as separate problems prior to their discovery. The discovery of a critical number was usually not a sudden event, but a gradual process of isolation and of growth of meaning. The process will be referred to hereafter as the abstraction of critical numbers. Various stages of the process must be distinguished and defined before quantitative comparison becomes possible. The first stage in the abstraction of a critical number may be regarded as completed when, with this number of beads concretely before him, the subject gives the first clear indication of his realization that in the current trial defeat is inevitable. This recognition of defeat may be expressed in a shaking of the head or in some form of exclamation, such as, "I lose," "No use," "You win," "I cannot win now," or "No matter how I draw now you can win." Such recognition of defeat often comes in the form of a perceptual judgment, without explicit awareness of the exact number of beads remaining or of the impossibility of ever winning from this particular number. The second stage in the abstraction of a critical number will be regarded as completed when the subject clearly states the recognition of his inability *ever* to win the number in question. The third stage is regarded as completed when the subject announces his conviction that the number in question is also critical for the experimenter; i.e., that the experimenter must inevitably lose if required to draw from that number. In the interpretation of

reactions indicative of the various stages the general context must be taken into account; our criteria are not, therefore, wholly objective.

The individual records of twelve new subjects, reacting to 14 beads as their initial problem, are given in Table VIII. The subjects are listed at the left as S1, S2, S3, etc., according to the number of trials required to win two trials in succession from 14 beads. In the *f* columns of the various A sections (i.e., 3A, 6A, 9A, and 12A) of the table are given the numbers of draws made by individual subjects from each of the critical numbers during the first stage of its abstraction. Similar data for the second and third stages of abstraction are given in the B and C sectons of the table respectively. The totals of draws from the two non-critical numbers adjacent to each critical number, during the various stages, are given in the *s* columns of the respective sections. Thus the record of S3 shows that only the first critical number was abstracted. During the first stage, 3A, of this process two draws were made from 3 beads and none from 2 or 4 beads. Prior to the completion of Stage 3B, 39 draws were made from 3 beads and a total of only 6 from 2 and 4 beads. Before 3C was completed 74 draws had been made from 3 beads though the total number of draws from 2 and 4 beads was only 16.

In only 4 of the 73 pairs of *f*- and *s*-column figures here presented does the number of draws from a critical number fall short of the sum of all draws from the *two* adjacent non-critical numbers, and in no case does it fall below the average number of draws from the two adjacent non-critical numbers. The average number of draws from critical numbers at the completion of Stage A in their abstraction, is 31.2; that of draws from adjacent non-critical numbers is 4.2. At the completion of Stage B the corresponding averages are 51.6 and 9.9 respectively; and at the completion of Stage C they are 100.6 and 21.8 respectively. The total number of draws from all critical numbers in all stages of their abstraction as here reported, is 3787. From non-critical numbers—though there are twice as many of them as of critical numbers—the total number of draws is only 703. That is to

TABLE VIII
Frequency of Subjects' Draws from the Critical Numbers at the Completion of the various Stages of their Abstraction.

	3A		3B		3C		6A		6B		6C	
	f	*s*	*f*	*s*	*f*	*s*	*f*	*s*	*f*	*s*	*f*	*s*
S1	1	0	55	2	167	31	52	4	57	4	158	40
S2	4	1					7	1				
S3	2	0	39	6	74	16						
S4	1	0	28	4			3	0	14	2		
S5	1	0					4	0	20	0	43	14
S6	1	0							11	1		
S7	3	1			11	5	2	1	12	8		
S8									22	5	68	26
S9	1	0					27	2				
S10	2	1					76	15				
S11	7	0					19	0	66	20		
S12	7	1					7	1	90	9		
Total	30	4	122	12	252	52	197	24	292	49	269	80
Av.	2.7	.4	40.7	4.0	84.0	17.3	21.9	2.7	36.5	6.3	89.7	26.7

	9A		9B		9C		12A		12B		12C	
	f	*s*	*f*	*s*	*f*	*s*	*f*	*s*	*f*	*s*	*f*	*s*
S1	57	13	64	13	148	50	104	74			108	82
S4	15	2	21	5			17	8	22	10		
S5	24	12	27	12	33	24	32	40			33	42
S6			11	7			80	28	83	31		
S7			22	26	52	34	19	35				
S8	24	27	38	27	55	39	29	37				
S10	98	41	103	41	118	48						
S11	36	11	85	70			60	51				
S12	105	11	125	11	251	45	164	149	172	149	190	160
Total	359	117	496	205	657	240	505	422	277	190	331	284
Av.	51.3	16.7	55.1	22.8	109.5	40.0	63.1	52.8	92.3	63.3	110.3	94.0

say, at the time of completion of the various stages of abstraction of the critical numbers, the number of draws from them is on the average more than ten times as great as from adjacent non-critical numbers.

The order of abstraction of the various critical numbers is also in practically complete harmony with the order of frequency of subjects' reactions to them individually. Not only are a subject's draws from a lower critical number invariably more numerous than from a higher one, but his draws from the lower non-critical numbers are also fewer than from higher ones. Both of these facts tend to give greater emphasis to the lower critical numbers and favor the abstraction of critical numbers from lower to higher in the order of their magnitude. A comparison of the actual order of abstraction requires that different stages of the process be listed separately. This has been done in Table IX.

As before the subjects are listed at the left. The critical numbers appear at the top of the columns, the various stages being segregated as indicated by the capitals. The figures in the columns indicate the order in which the critical numbers emerged from the various stages of abstraction. Thus the record of S1 shows that 3 was the first number to emerge from Stage A, 6 the second, 9 the third, etc. Points where hesitation and other inconclusive signs showed the probable occurrence of abstraction though no clear indications of the process were observed, are marked with asterisks. No overt indications of abstraction were observed for the blank points in the table. When a given stage in the abstraction of two or more critical numbers was completed simultaneously, they were given the same value in the table.

Table IX

Order of Abstraction of Critical Numbers.

	3A	6A	9A	12A	3B	6B	9B	12B	3C	6C	9C	12C
S1	1	2	3	4	1	2	3	*	1	1	1	1
S2	1	2	–	–	–	–	–	–	–	–	–	–
S3	1	–	·	–	1	–	–	–	1	–	–	–
S4	1	2	3	4	1	2	3	3	–	–	–	–
S5	1	2	3	4	–	1	2	*	–	1	1	2
S6	1	*	*	2	–	1	2	3	–	–	–	–
S7	2	1	*	3	–	1	2	*	1	–	2	–
S8	–	*	1	2	–	1	2	–	–	1	1	–
S9	1	2	–	–	–	–	–	–	–	–	–	–
S10	1	2	3	–	–	–	1	–	–	–	1	–
S11	1	2	3	4	–	1	2	–	–	–	–	–
S12	1	2	3	4	–	1	2	3	–	–	1	1

That the correspondence between the frequency of reaction to various critical numbers and the order of their abstraction is really very close, is shown by the fact that in only one case (S7, 3A) in the entire table did a lower number emerge later than a higher one from a given stage of abstraction. There is, however, a strong tendency in this group for all critical numbers to emerge simultaneously from Stage C. This is due to the fact that abstraction has here reached a sufficiently advanced stage to permit generalization to become effective.

b. Effect of Nearness to a Goal.—As already noted the frequency of reaction to the various critical numbers varies directly with their nearness to 0, the immediate goal of every trial. It is therefore impossible to measure separately the effects of fre-

quency of repetition and those of the nearness of elements to a goal. Logically the relative nearness of elements to a goal resolves itself into two distinct factors: (1) the greater ease with which the consequences of tentative or hypothetical draws can be foreseen from lower than from higher numbers and (2) the closer temporal proximity of the goal to reactions toward lower than to those toward higher critical numbers. Both of these factors were probably operative as will later be pointed out more fully. Some evidence of the effects of (1) is found in the percentages of erroneous draws from various non-critical numbers as reported in Table VI. In the further presentation of evidence at this point no effort will be made to distinguish between the effects of these two factors.

That the critical character of 6 is more easily discovered than that of 9, is shown by the number of trials required by 24 subjects from each of these numbers in discovering the impossibility of winning it. The total numbers of trials upon 6b and 9b were 126 and 194 respectively. Seventeen of these subjects required fewer trials upon 6b than upon 9b. Likewise fewer trials were required upon 9 than upon 12 beads, both by the group as a whole and by a majority of the individual members. But, owing to the rapidly increasing effects of previous learning upon the higher numbers of the series, the difference here is not so great as in the former pair.

The comparative difficulty of abstraction of various critical numbers is further shown in Table VIII. The averages as they stand show a marked general increase in the number of reactions required to raise successively higher critical numbers out of a given stage of abstraction. But this evidence is open to the objection that the averages do not all represent the work of exactly the same group of subjects. The results are not much modified, however, when the work of those subjects whose records do not show complete data for the stages compared, is left out of account. In the work of 9 subjects whose records contain data for both 3A and 6A there is only one case, that of S7, in which 3A offers more difficulty than 6A. The average numbers of draws are 3 and 21.9 for 3A and 6A respectively. Stages

6A and 9A are both represented in the work of 6 subjects all of whom required more draws from 9 in the attainment of 9A than from 6 in the attainment of 6A. The average numbers of trials are here 26.7 and 55.8 for 6A and 9A respectively. Data for both 9A and 12A are contained in the records of 6 subjects all of whom required more draws for the attainment of 12A than of 9A. The average numbers of draws are here 43.5 and 67.7 for 9A and 12A respectively. Without going into further detail it may be said that the results in the B Stages show the same marked tendency for higher critical numbers to require many more reactions for their abstraction than are required by the lower ones. The less marked tendency in this direction in the C Stages is clearly due to the onset of effective generalization and need not concern us here.

c. Effect of Speed of Reaction.—The ease with which the principle of drawing by "opposites"[22] is discovered appears to depend largely upon the speed of reaction of the subject. The subjects who reacted rapidly almost invariably discovered the "opposite" relation between their draws and those of the experimenter early in the game, but those who reacted slowly usually became aware of this relation late or not at all. Data regarding the speed of reaction and the number of trials required by subjects of Groups I and III are presented in Table X.[23] The data for Group I are shown in the upper section of the table, those for Group II in the lower section. Since the extent of the series is not the same for different subjects, early or late discovery of the principle of drawing opposites must be determined not by the absolute number of trials required for the discovery, but by the percentage of all reactions upon Series 1-2 which sufficed for the discovery of the principle. These percentages are given in the last column of the table.

[22] That is, drawing 1 when the opponent draws 2, and 2 when the opponent draws 1. This procedure obviates the necessity of counting the remaining beads when their number has once been reduced to a multiple of 3.

[23] Similar data cannot be given for Group II owing to the fact that no separate records were kept of the time required for the solution of individual problems of the series.

Table X

Speed of Reaction as Related to Ease of Discovery of the Principle of Drawing by "Opposites."

Subjects	Av. No. of seconds per draw in discovering the principle of opposites[24]	No. of trials required for discovery of the principle of drawing by opposites	Per cent of total trials upon Series 1-2 required for the discovery of the principle of drawing by opposites
i	10.25 (5.2)[25]	—	1.00
ii	12.28 (12.8)	—	1.00
iii	5.94 (5.2)	25	.89
iv	5.74 (6.0)	25	.26
v	12.76 (6.4)	—	1.00
vi	2.86 (3.4)	71	.21
vii	5.05 (4.0)	21	.06
viii	3.67 (3.1)	167	1.00
ix	4.88 (4.5)	308	.96
x	4.56 (6.1)	77	.13
xi	6.07 (13.5)	—	1.00
xii	4.98 (5.1)	70	.38
xiii	2.59 (3.1)	86	.52
xiv	3.18 (5.7)	88	.24
1	6.88 (5.8)	32	.57
2	7.49 (5.3)	65	1.00
3	13.13 (11.6)	—	1.00
4	3.96 (7.2)	—	1.00
5	4.85 (4.4)	—	1.00
6	2.70 (3.1)	22	.31
7	6.16 (5.8)	35	.26
8	6.08 (5.6)	107	.75
9	5.13 (5.5)	—	1.00
10	3.83 (3.9)	63	.24
11	3.57 (4.1)	111	.39
12	3.78 (3.1)	84	.24

[24] Or until the completion of Series 1-2.

[25] In the parentheses beside the second column the average number of seconds per draw in the first fifty draws of each subject, is given.

Note the general correspondence between the ease of discovery, as indicated by the percentage values in the last column, and the speed of reaction as indicated in the second column of the table. The average time per draw for subjects who did not discover the principle of drawing opposites was 10.34 seconds in Group I and 6.77 seconds in Group III. The averages for subjects who did discover the principle were 4.35 and 5.06 for Group I and Group III respectively. By the rank method the correlation between the ease of discovery of the principle of opposites and the speed of reaction is .668 for Group I and .516 for Group III.

It might be supposed that these correlations are due to an increase in the speed of reaction after the discovery of the principle of opposites but before its announcement. That the correlations were perhaps raised somewhat by this tendency is not denied. But the effect of this factor was probably slight since the subjects were urged repeatedly to notify the experimenter of any new discovery or seemingly relevant new idea at the time of its occurrence. Moreover, the records show that subjects who reacted slowly at the beginning of the work usually failed to discover the principle of drawing opposites, whereas subjects who drew rapidly from the beginning were likely to discover the principle early in the game. Thus the average time per draw of the first fifty draws of Subjects i, ii, v, and xi, who did not discover the principle, is 9.47 seconds, whereas the average time of the remaining subjects of this group, all of whom discovered the principle, is only 4.42 seconds. The corresponding averages for subjects of Group III who did and for those who did not discover the principle, are 7.20 and 4.60 seconds respectively. The correlation between ease of discovery of the principle of opposites and the speed of reaction in the first fifty draws is found to be fairly high. For Group I this correlation is .373 and for Group III it is .629.

It may still be argued that the speedy reaction of some subjects was merely the result of their attention to the relation between their own draws and those of the experimenter; but such comments of subjects as we have, together with their time records, show that even while attending to the number of beads remaining or recalling their success or failure with certain numbers in past trials, these subjects reacted with less deliberation than others. As shown by their comments, the work of these subjects was characterized by incessant fluctuations of attention between relatively isolated elements or by wholly irrelevant reflections. These subjects appeared to react quickly not so much because of any marked speed of mental activity as because of a lack of those deeper insights into relations and consequences, which might have barred hasty reaction.

In striking contrast with these positive correlations between

speed of reaction and relative ease of discovery of the principle of drawing opposites, are the correlations between speed of reaction and facility in finding a solution for the entire series, as measured by the number of trials required for the solution of the series. The average speed of reaction in the entire series for all subjects of Groups I and II, together with the number of trials required for the solution of the series, is shown in Table XI.[26]

Table XI

Speed of Reaction as Related to the Number of Trials Required for Solution of Series 1-2.

Subjects	Average No. of seconds per draw in Series 1-2	No. of trials required for the solution of Series 1-2
i	10.25	50
ii	12.28	35
iii	5.52	28
iv	6.89	96
v	12.76	62
vi	3.19	331
vii	4.00	371
viii	3.67	167
ix	4.60	315
x	4.33	597
xi	6.07	53
xii	6.54	184
xiii	3.68	166
xiv	2.77	367
A	12.22	48
B	7.00	62
C	5.61	85
D	11.86	112
E	6.70	220
F	11.67	300
G	4.68	322
H	7.00	389
I	4.86	617
J	3.91	823

The data for Group I are given in the upper portion of the table and those for Group II in the lower portion. The correlations between speed of reaction and the number of trials required for the mastery of the series are —.748 for Group I and —.686 for Group II, when the speed of reaction is stated in

[26] Similar data are not available for Group III owing to the failure of most of the subjects of the group to obtain a solution for the series while solving the limited number of problems presented to them.

terms of the average time per draw in the entire series. When the speed of reaction is determined on the basis of the first fifty draws of each subject, the correlation is —.264 for Group I. Thus intelligence as determined by the ease of solution of Series 1-2, does not affect the speed of reaction so much at the beginning of the series as later when there is a greater accumulation of experience which may serve to inhibit hasty reaction. Speed of reaction appears then in the double rôle of an effect of the depth of insight into the conditions of the problem and as a factor in determining the direction of analysis.

2. *Explicitness and Extent of Analysis*

In the analysis of puzzles Ruger reports a "wide variation of felt clearness from extremely vague to perfectly clear. This range of felt clearness," he finds, "is matched by differences in results." Our results show a similar variation in the explicitness of analysis from cases in which an element of a situation is barely recognized in passing, to those in which clear verbal formulations are accompanied by ability to recognize and control the element in question under novel conditions. Numerous cases were observed where analysis was complete enough to be effective for manipulation but not clear enough to be put into words. A great many of the perceptual solutions were of this character. The analysis of the lower numbers practically never reached the explicitness of a clear verbal formulation until generalizations made upon higher numbers were applied to them. Occasionally a subject would declare that he knew how to win a given number but could not express it in words. One subject, after winning from 20 beads twice in succession declared that she believed she could do it again though she could not say how. She then proceeded to win two more trials in succession without difficulty. Another subject, having just won twice in succession from 10 beads, was asked whether she had any new ideas. She replied: "I can't state the idea in words but I have it in motor terms."

Ruger mentions several different sorts of analysis which affected only a part of the situation. A very indefinite type of partial analysis he describes as "picking out the portion of the

puzzle to be attacked. In many cases," he says, "there was a mere spatial analysis, 'locus analysis' without involving any perception of the mechanical necessities concerned." This locus analysis resulted in the confining of random movements to a portion of the puzzle. This is very similar to a common experience of our subjects many of whom quickly discovered that the crucial point of the trial lay in the early draws, and busied themselves, accordingly, with the task of determining what the first one or two draws must be. This often resulted in some impatience with the requirement that every trial must continue until the number of beads was reduced to 0, and necessitated a few brief departures from the rule. Often the point conceived of as crucial was in the lower and sometimes in the intermediate numbers rather than in the higher ones, as the following examples show:

"I wonder if you often draw 2 at the beginning as I do, merely to hurry and reduce the number of beads" (xiv, 23b).[27]

"No matter what number you are working with you can win if you can get it to 4 or 5 beads. The problem isn't with the larger numbers but with the few at the end" (xii, 11b).

"It seems to revolve about the start" (xii, 16b).

"Well, I would eliminate the beads until I got my first turn on 4 or 5 (xii reacting to the problem of 59 beads).

"When I win, the crucial draw is the second, or not later than the second" (xiv, 25b).

A slightly more definite type of analysis is described by Ruger as follows: "An important form of partial analysis noticed was that of a single step in the process while the other steps were attained only by random movement. This single step was often the final one. The solution would come accidentally but the subject would notice the last step. In subsequent trials he would know what to do if he chanced to get to that step but not how to get there." The folowing comments from subjects indicate this type of partial analysis:

[27] The Roman numerals in parentheses indicate the subject whose comment is quoted and the numbers following show the particular problem in the series, during the solution of which the observation occurred.

"I can't beat when we get to 6. I never have beaten you on 6" (xiii, 12b).

"I can't win this; there are 6 left" (ix, 11b).

"I could win if it would come out 11 there" (ix, 14b).

"I know I can't win unless I can make it 13 there . . . after you have drawn" (vi, 25b).

"Some of the first moves I win and others I lose. Now on 5 if I have the first move, I win" (vii, 22b).

"I was trying to get it so there would be 7 beads left on my draw" (viii, 12b).

These partial analyses, especially when concerned with the higher numbers in a series, depended largely upon the possession of some general scheme by means of which attention could be freed from most of the details of the series and concentrated upon some one point. Such schemes usually took the form of drawing uniformly either 1 or 2 beads throughout the trial, and generally resulted quickly in some sort of appreciation of the principle of drawing by opposites. This principle is one of the best examples found here of what Ruger called "schematic analysis." A partial insight into the significance of multiples of 3 sometimes served as a schema of this sort. One subject, for example, expressed a schematic view of the line of approach to a solution of Series 1-2 as follows: "You draw 2 when I draw 1, and 1 when I draw 2, so that we reduce it by 3's each time; that is, we are drawing them out by multiples of 3" (vii, 18b). From this comment it might be thought that the subject had effected a complete analysis of the series, at least up to and including 18 beads; but it required two additional trials to bring him to realize that he could not win 18 beads, and many more trials to complete the series. During the next trial he observed: "We are drawing right down on multiples of 3"; but here again he failed to realize the significance of multiples of 3, for it required still another trial to convince him that he could not win 18 beads, and his conclusion was not based on multiples of 3 at all. "I can't win from 6," he said, "so since this is a multiple of 6, I can't win it." Of the problem presented by 18 beads the subject had probably made a total analysis in the sense that it

"reached all the elementary steps or movements." That it was far from a complete analysis of the entire series is evident from the fact that it required still 45 trials upon 19, 20, 21, and 22 beads for the development of sufficient insight to permit the subject to give a solution for the entire series. At this stage of the process, however, analysis proceeded far less than formerly by frequency of repetition, nearness to a goal, etc., and far more by means of ideas and principles formulated in early problems and now applied to new problems and verified with a minimum of repetition. In other words, generalization, which has played a rôle of increasing importance since the earliest stages of analysis, has here become the dominant factor.

3. *Time Relations of Manipulation and Analysis*

In describing the time relations of ideational analysis and motor variations Ruger says: "These two types of variation, acts of analysis and motor responses, may be quite varied, especially in their time relations. At the one extreme is the motor variation which, perhaps, brings success but which runs its course unnoticed. At the other extreme the analysis may occur first and only after a considerable interval be followed by the motor response."[28] This wide variation in the time relations of analysis and manipulation is very characteristic of the work of our subjects, except that in the first series we find little ideational analysis preceding manipulation. Our procedure was not of course such as to encourage attempts at analysis prior to manipulation in the early problems of the series, though we believe that such efforts were seldom discouraged except by their failure to bring results. Such analysis as occurred early in the series was usually of a perceptual nature, as already explained, and was accompanied and checked by manipulation of the beads. The extent to which the method of trial and error became operative in the intermediate and higher numbers of the series, is indicated roughly by the percentage of erroneous draws. These percentages for some of the subjects in the first six problems of the series are given in table V. In approximately the first two-

[28] *Op. cit.*, p. 12.

thirds of the trials of a subject upon each of the higher numbers the erroneous first draws usually amounted to about 50 per cent of all of the draws from the number in question. Often, indeed, half of the first draws of all trials in a number of successive problems, were erroneous. Thus Subject viii, in working from 7 to 14 beads in the first series, made 86 errors of this sort out of a possible 175. In the 7 problems from 8 to 14 beads inclusive the possible errors for this subject were 156 and his actual errors were 86. Subject i required 38 trials in working over the numbers from 7 to 14 inclusive in the first series; i.e., there were 38 possibilities of erroneous first draws. Although this subject was one of the most successful of the lot and made special effort at analysis before manipulation, he made 21 errors in the 38 draws. His comments upon his methods of procedure are of interest here: "I'm doing this by the trial and error method until I get it down to 4 or 5 where I can handle it." Later he says: "I find that any attempt to analyze this series is apt to inhibit action; at least it doesn't conduce to a solution." Repeated comments of this nature were made by many of the subjects.

It often happened, while drawing through a trial which had been given up as lost, that success would be accidentally achieved. Frequently the subject would attempt to recall how it was done. Often he succeeded, but even if he failed, the effort was not entirely lost, for the arousal of attention to the existence of neglected possibilities resulted in a change of attitude and sometimes in a fruitful examination of his assumptions. Numerous occurrences of accidental success without ability to recall the successful variation were noted. The following case is representative:

Subject ix had experienced particular difficulty with 11 beads, Series 1-2, and finally concluded that he could not win if the experimenter drew 1 early in the trial. He was persuaded to continue, however, and finally happened to take 2 at every draw and so won the trial. His comment was: "I just fell on to that by trial and error procedure. It was accident; I don't believe I could do it again." Some time afterwards he met with the same

difficulty when drawing from 14 beads. After winning with considerable difficulty, he said: "I don't know how I did that." Later on with 17 beads he repeated his old error seemingly none the wiser for his two accidental successes.

A striking example of how accidental successes may be seized upon and made the basis for further progress is found in the reactions of Subject vi who, having made good progress up to 25 beads in Series 1-2, became stranded and required 156 trials to win this number twice in succession. This subject soon discovered that when his initial draw was 1 and the experimenter drew 2, he could win by taking 1 at every draw throughout the trial. But when the experimenter took 1 at the first draw, the subject was helpless, though he could have won by merely changing his draws to 2 from this point to the end of the trial. Somehow the latter procedure escaped his attention, although he followed it once by accident near the beginning of the plateau, so that with 57 opportunities to win by this procedure he succeeded only twice. After his 155th trial upon 25 beads, when the successful variation occurred for the second time, he said: "That's another way I win—certainly I could win that way if I won the other way!" When questioned as to whether he had known from the beginning of the trial that he would win, he replied: "Not until the third to the last draw. In fact, I started drawing 2 without any intention of continuing to draw 2 on down through the trial. Then I later decided to continue (because, as subsequently brought out in a comment, he did not remember having tried that before) as I started. . . . It didn't occur to me until after I had finished, that it didn't make any difference whether I got that one extra bead first or last, if you drew 1 each time."

The last sentence throws some light on the nature of the difficulty experienced by him with this combination. If the subject and the experimenter drew 1 bead each at the first draw, there would be 23 beads remaining. Now, if the subject took 1 bead at his second draw and the draws proceeded by opposites after that, there would be 2 beads left for him at the last draw; that is, "1 *extra* bead." The subject's reactions clearly show that some such assumption as, that he must provide for that "extra

bead" at his first opportunity, was made; for of the 55 errors which he made in drawing 1 when he should have taken 2, forty consisted in drawing 1 from 23 beads. The other 15 errors of this sort can be accounted for largely on the basis of his efforts to reach certain lower numbers which he was gradually learning to regard as significant. This false assumption explains why the subject made 40 errors in 57 draws from 23 beads. The case shows how an accidental success may serve to clear away false assumptions and so contribute indirectly as well as directly to a more speedy solution of the problem.

It is worthy of note that the accidental successes in the foregoing cases, which occurred before a fair acquaintance with the series had been formed, either failed to attract attention or could not be recalled, whereas an accidental success occurring later in the series was quickly seized upon and applied to the solution of the series. This was quite a common occurrence in the work of a majority of our subjects.

Having seen the futility of early attempts at analysis without manipulation, and the varying degrees in which subjects profited by accidental success at different stages of their mastery of a series, we turn briefly to the growing capacity for ideational analysis, which usually began to be effective towards the end of Series 1-2 and became increasingly apparent as the experiment progressed. The first successful attempts at ideational analysis of more than perhaps one step in the series usually consisted in a return to lower numbers which were regarded as related to the present situation, with an attempt to determine their forgotten status, or in a return to the very beginning of the series to recover orientation. Reasoning about these lower numbers would invariably take the form of ideational manipulation mentioned earlier in the discussion. Usually some numbers remembered as critical served to facilitate the determination of the status of other numbers. An example will show the character of this sort of analysis more clearly:

After gaining a fair acquaintance with Series 1-2 in 371 trials upon the first 17 problems of the series, Subject vii took an excursion back to the beginning to check up and get his bearings.

"Some numbers," he said, "the first (draw) wins and some it loses. Now at 5 beads if I have the first move, I win; at 7 I lose—I take 2 and you take 2 and I lose. From 7 if I take 2, that leaves you 5 and you can take 2 and win. If I take 1 and leave 6—if I have 7 and take 1, I win. If I have 9, I lose. If I have 9 and take 1, I lose the game; if I have 9 and take 2, I lose, therefore I lose on 9 always. If I have 8 and take 1, the other fellow wins; if I take 2,—O yes! *I see why it is that I win 8, because I leave 6.* I can win 5 by taking 2 and I can win 7 by taking 1; 6 I can never win. If I have 8 and take 2, I always win. I can win 5 if I take 2; 6 I can never win, 7 I can win if I take 1, and 8 I can win if I take 2; 9 I can never win—6 and 9 I can never win; 10 if I take 2, I lose; if I take 1, I can always win. 11 if I take 1, I lose; if I take 2, I can always win. 12, if I take 2, you can take 1 and leave 9; if I take 1,—I can't win it. 5 I win and 6 I lose—on even numbers I take away 1. On 11 if I take 2—12 I lose because you can leave either 9 or 8. 6, 9, 12 I can't win. I can't win any multiple of 3. On other multiples of 5—I can't carry it out. Let's go on."

Here the subject was asked how he would draw from 59 beads. "An odd number," he said. "This will be merely a guess. I should judge that since it is an odd number, I could win. I would try to leave it odd all the way. I would take 2 and the other fellow couldn't win if I watched my draws. I think he couldn't win if I watched my draws. I think he couldn't win multiples of 3—*I would reduce him to multiples of 3!*"

Even after the solution of an entire series most of the subjects showed but slight tendency towards or capacity for anticipatory analysis in the attack upon new series. Twelve of the fourteen subjects in Group I began the attack upon Series 1-3 in exactly the same manner as that upon the first series; that is, they began to draw without delay and only after some time attempted to formulate a statement for the series. The effect of their work upon the preceding series was very noticeable, however, in the readiness with which some subjects took note of the fact that 4 could not be won, and in their alertness for other critical numbers.

Subject i said at the beginning of Series 1-3: "I am going to try to reduce this trial and error to a minimum. With 1 or 2 (i.e., in Series 1-2) the smallest combination was 3; 1, 2, or 3 is a more difficult proposition. I imagine I could sit down and work it out." He drew 2 and the experimenter took the remaining 3. "Oh, I forgot," he said "that there were three possibilities. I will try all the possibilities now and see if you can get me." From this point manipulation and analysis progressed together. This failure of one of the most successful subjects, certainly the one who was most inclined towards anticipatory analysis, shows the difficulty of such analysis in the early stages of acquaintance with the elements of the problem. His progress had been, however, very rapid in the first series so that time had not permitted a very thorough stamping in of the elements, and he had altogether failed to notice some of the important elements of the series.

When presented with Series 1-3, Subject xiv began to draw after slight hesitation but said immediately after the first trial: "Now I think I can get this right off. This time you are going to let me win every fourth one; that is, on the fourth you *won't* let me win and on the fifth you *will* let me win, provided I start with 1. On 6 I should have to start with 2, and with 3 on 7. (And on the 8th?) I wouldn't win 8." This subject quickly solved two more series of this order without a draw.

This exceptional case may well be due to the fact that Subject xiv made a more thorough analysis of Series 1-2 than was effected by any other subject. Also by a great deal of shifting of attention from one set of elements to another during 367 trials occupying 8173 seconds, as compared with an average of 204 trials in 4943 seconds for all subjects of Group I, he had probably succeeded in stamping in these elements more thoroughly than had been possible for most of the subjects. With less than half as many trials as xiv required for Series 1-2, Subject i solved in the neighborhood of thirty series of problems and gave satisfactory generalizations for all problems included in the experiment. It is clear, therefore, that the insight shown here by Subject xiv cannot be regarded as a case of ideational analy-

sis without a fair acquaintance with the elements of the situation.

It required the solution of only one or two additional series of problems to give this control of all series of the first order (i.e., all series in which L is equal to 1) to nearly all of the subjects, and all succeeded in getting a general solution after only a few more series. But no subject solved Series 2-3 without manipulation of the beads. The solution of fewer series was required, however, for the mastery of the second order of series than for the first order. Likewise the mastery of all series of the third order required actual work on fewer series than that of either of the preceding orders, and usually resulted in the mastery of all series in which the numbers between H and L might be drawn. This transfer of power from series to series was due largely to the easy recognition of certain elements of the new series, which were identical with familiar elements of old ones, thus obviating in part the necessity for new analytic activity. The transfer of power was further facilitated by generalizations upon familiar elements, which were easily applicable to new series. These factors of familiarization and generalization became more prominent as the work progressed, but in series where some new elements were introduced there was still a ncessity for genuine analytic learning. Much of this analysis continued to follow or accompany manipulation, but a gradually increasing proportion was performed in ideational terms, largely by the sort of ideational trial and error already described. Thus with increasing acquaintance with the materials we find a continual receding of activity from overt trial-and-error manipulation to a very similar sort of process carried on in ideational terms, which in turn gives way to general ideas that have evolved gradually by the bringing together of similar elements in association with general symbols. In the later series all three of these factors were almost constantly in operation.

4. Summary

After the range of perceptual solutions was passed various elements of the problematic situation were gradually abstracted and associated with verbal symbols. The type or combination

of types of elements so abstracted differed largely for different subjects and in different periods of the work of the same subject. The type of elements abstracted was shown to vary in accordance with the speed of reaction. Frequency of reaction to various elements and their nearness to a goal, were shown to be closely correlated with the order of their abstraction. Large differences in the degree of explicitness of analysis were observed, and there was found to be a somewhat gradual development from vague to clear and explicit states of analysis. The extent of analysis varied with different subjects and at different stages of the learning process, but analysis usually occurred first in isolated spots and spread from these to other portions of the series. The time relations of ideational analysis and manipulation varied greatly. In general, manipulation preceded analysis in the first few series, and persisted throughout the greater portion of the learning process as an important method of procedure. Gradually, however, as acquaintance with the elements of the situation grew, overt trial and error was replaced by a very similar sort of ideational manipulation which, in turn, tended to give way to general ideas.

E. Generalization

In the preceding section attention was directed principally to the abstraction of the elements of the problems presented for solution. As was shown in several instances, however, such elements as are abstracted do not remain in their early state of relative isolation during the entire course of analysis, but tend to combine into higher units which become associated with appropriate symbols and take on general meanings. This process of generalization and the resulting general ideas will be the subject of discussion in the present section.

1. *Relative Absence of Generalization in the Perceptual Stage*

Mention has already been made of the apparent lack of generalization during the period of perceptual solutions. Practically no comments were made during this period, which would seem to indicate any attempts at generalization. Moreover, when a

subject did finally attempt to generalize, those numbers which are well within the range of perceptual control were rarely taken into account, though they had been more often repeated than any of the higher numbers. Only two of the fourteen subjects of Group I noticed, with sufficient clearness to mention the fact, that 3 is a critical number, until attention was directed to it by their final generalizations which were based upon higher numbers. Some of the subjects were very much surprised that this fact had so long escaped their attention. The first important landmark mentioned by eleven of the subjects of this group was 6, and this was generally mentioned not earlier than the last trial on 9 beads, sometimes not until much later.

But this failure to recognize the status of lower numbers in the abstract does not preclude recognition of the critical or non-critical character of the concrete situations for which they stand. In fact, the situation which is represented by the critial number 3, was almost invariably the first to be reognized in the concrete as critical. This fact is shown in Table VIII where the realization of the impossibility of winning 3 is shown to have been the first step taken by ten of the twelve subjects of Group III in the abstraction of critical numbers. Table IX, however, shows that 3 was explicitly mentioned as a critical number before 6 was so mentioned, by only 2 of the twelve subjects. The failure of subjects to utilize numbers below 6 along with higher numbers as a basis for generalization, is not, therefore, due to the difficulty of recognizing their status in the concrete but rather to the ease of such recognition, which made it unnecessary to associate these situations with verbal symbols.

2. *Development of the Concept of the Critical Number*

Though the concept of the critical number seems to be a very simple affair, its development required the expenditure of a considerable amount of time and effort on the part of the subject. To facilitate the description of this development we may divide it into the following seven stages:

Stage A.—The first appearance in explicit form of the critical-number idea was at the point where the subject discovered

that he could not win 6 beads. But this was straightway forgotten in most cases, or at any rate failed to function in slightly new situations. Similarly, the impossibility of winning 9, 12, etc. was later discovered without a full realization of the significance of these numbers at the time. This first discovery of the impossibility of winning a number constitutes the first stage in the growth of the concept of the critical number. Critical numbers in this stage stand out in comparative isolation.

Stage B.—Since the subject was required to draw first in every trial, it was impossible to compel him to draw from a given critical number in any trial in which the inital number of beads was only 1 greater than the crtical number in question. Therefore during the solution of the next higher number in the series the recently discovered critical number was usually forgotten. Its re-discovery after more or less delay constitutes the second stage in the development of the concept. Here the individual critical number is recognized as an element of all problems of the series; e.g., 6 is recognized as critical regardless of what may have been the initial number of beads presented in the trial.

Stage C.—At this stage the relationship between two or more critical numbers in Stage B, or perhaps in Stages A and B, was discovered; as, "I cannot win 9 because you can always reduce me to 6"; or, "I cannot win 12 because I lose 6, and 12 is composed of two 6's," etc. Here the critical numbers are not only recognized as critical in all problems of the series, but they are associated with each other in more or less definite relations.

Stage D.—The fourth stage consists in the discovery of the fact that the subject cannot win any multiple of 3. All critical numbers are recognized as critical and all are associated with a common symbol.

Stage E.—The foregoing stages all deal with the critical numbers as related to the subject's recognition of the possibility of success. A somewhat similar development may be observed in his utilization of the critical-number idea as a means of control in the achievement of success. Thus it often happened that a subject discovered the possibility of winning by reducing the number of beads to 6 and so forcing the experimenter to draw

from it, long before learning that the same procedure applied to the higher critical numbers would bring the same results. In this stage one particular number is recognized as critical *for the one who draws first.*

Stage F.—In Stage F two or more critical numbers are recognized as having this broader significance.

Stage G.—Here all critical numbers are recognized as critical for the one who draws first.

It must not be supposed that learning progressed with perfect regularity from stage to stage in exactly the foregoing order, or that all of the stages were distinguishable in the progress of every subject. They do, however, occur separately in the majority of cases, and in some instances in the exact order named. The following comments of subjects will serve to illustrate this development. They are all taken from Series 1-2. The stage of development of the concept, as inferred from each comment, is indicated in parentheses. The portion of the series in which the comment occurred is in each case indicated at the left. 12b, 12, for example, signifies that the first comment occurred during the twelfth trial on 12 beads.

Subject vii

12b, 12:	"You always manage to leave me 6." (B)
" 26:	"I can't win 12 because you can always reduce it to 6; 12 is two 6's." (C)
15b, 11:	"You reduce it to 6 every time." (B)
" 22:	"You always manage to get to 6; I don't know how I am going to prevent you." (B)
" 24:	"There are 12 left; I can't do much with that." (B)
" 24:	"There are 9 left; I haven't been able to win any 9's." (B)
18b, 8:	"Evidently this is one that you don't expect me to take because you throw it into the 6's." (C)
" 9:	"Your idea is to balance my moves so as to make it 6; and 9 is just as good for you." (B)
" 15:	"You draw 2 when I draw 1 and 1 when I draw 2, so that we reduce it by 3's; i.e., we are drawing them out by multiples of 3." (C)
" 16:	"We are drawing right down the multiples of 3." (C)
" 17:	"This is a multiple of 6; I can't win from 6 so I can't win it." (C)
20b, 1:	"The subject draws two and is asked why. "I have no reason," he replies. E then draws 2 and the subject takes 1, explaining: "I draw that way to keep it on multiples of 5 rather than on multiples of 3, and so to avoid 12, 9, and 6." (C)

" 2: After drawing 2 S says: "I can't win with 18." (B) (E?)

22b, 9: "If I have 8 beads and take 1, the other fellow wins. If I take 2—Oh yes, I see why I win: because I leave 6. (E) . . . I can't win any multiple of 3. . . . I can't carry it out; let us go on." (D) The subject is here asked how he would draw from 59 beads. After some irrelevant speculations he concludes: "I would take 2 and the other fellow couldn't win if I watched my draws. I think I would reduce him to multiples of 3." (G)

Subject xiii

10b, 6: "It depends on which draws first." (E?)

12b, 30: "I can't win when we get to 6. I never have beaten on 6." (B)

" 36: "I couldn't win 6; I don't believe I can win 12. I have been trying to see how to prevent you from leaving me 6. Double it and it is the same. It would be the same on all multiples of 6." (C)

15b, 6: "I can't beat you on this; it goes by 3's. *I have forgotten whether I beat you on 3.* 6 is the first I fell down on, isn't it? I don't know about 9." (D)

20b, 2: "I can't beat you on 16—Oh no, it was 18 I lost. (B) If it is 20 and I draw first, I win; but if you draw first, I lose. Let me see, I didn't beat you on 18. If you draw first on 18, I win." (F)

24b, 2: "I don't believe I can beat 18, 12, or 9, or 6. It must be multiples of 6 I can't win." (C) (D?)

26b, 2: "When I get 26, I see that by drawing 2 first I can give you 24, and I can therefore win." (F)

27b, 2: "I don't believe I can win. You make me draw on 24 no matter how I move. It must be multiples of 3, i.e., 3, 6, 9, 12, 15, 18, etc. *That's what it is!* (D) It depends on who draws first. There are certain numbers that the one who draws first can win and others that his opponent can win. (G) . . . After your first draw, i.e., after you get it to that number, draw opposites." The subject is here asked how he would draw from 59 beads and replies: "I would take 1 and, provided you take 2 every time—let me see, there may be another element there. I must draw 1 first on the odd numbers which I can win and 2 on the even numbers that I can win. But I am not certain yet."

28b, 3: "It looks like it is even numbers that I draw 1 on; and on odd numbers I draw 2 perhaps."

29b, 1: "That's the way it is. 3's I can't win, and multiples of 3. (D) 4 and 5 I win; 7 and 8 I win; 10 and 12 I win, etc. So on even numbers I draw 1 and on odd numbers I draw 2, and the opposite after the first draw."

31b, 1: "My scheme didn't work!"

" 2: "I have missed something there."

" 3: "It is a question of making you draw first on the multiples of 3. I didn't draw 2 here so that was wrong (i.e., his idea that he must draw 2 from odd numbers, etc.). It is a question of what to draw so as to make you draw first on a multiple of 3." (G)

Subject xii

9b, 0: "The first I couldn't win was 6, wasn't it? This is 9."

" 8: "Is it possible that with 6 or multiples of 6, such as 9 and 12, one can't get any results? that is, on all multiples of 3? (D) . . . In all cases of multiples of 3 I am going to assume that I can't get any results."

12b, 9: "I am positive I can't get this. I always begin with 6 (B). . . . *I came back to the original hypothesis* and thought of 6 as a multiple of 3." (D) S is here asked how he would draw from 59 beads. His reply shows that he has failed utterly to realize the significance of his idea of multiples of 3 for the achievement of success: "Well, I would eliminate the beads until I got my first draw on 4 or 5."

14b, 6: "When there are 6 and it is my turn to draw, you can always win." (B) The subject does not realize yet that 9 and 12 are also critical numbers.

14b, 9: "You can always make up the deficiency so as to take away an even number and leave 6." (B)

15b, 1: "I am going to try out that hypothesis again, that any number which is a multiple of 3 precludes any possibility of success for me." (D)

15b, 7: "Whenever a number is a multiple of 3, it is useless for me to try it." (D)

16b, 4: "When it gets to 9, I lose. (B). In this case I am working on the first 7 out of 16."

" 16: "It seems to revolve about the start."

" 37: "No matter how I move you can always bring me to 9. I lose 12 too." (B)

18b, 0: "I'll bet a cow I can't win 18." S is here asked how he would draw from 59 beads, and replies: "I couldn't get it."

21b, 1: "I lose multiples of 3." (D). When asked how he would draw from 61 beads, he says: "I would draw 2. (Why 2?) To get to 59 and bridge over 60. (E?)

22b, 1: "Well, I lose and I thought I had it cinched." The subject later stated that he took 2 beads at the first draw in this trial in order to "bridge over" 21.

22b, 2: "I see my mistake. I took 2 and you 2 leaving 18, another multiple of 3." (D)

24b, 0: "I'll pass it up." (D)

25b, 1: "That was foolish. I took 2 and allowed you to reduce it 21, a multiple of 3, and I lose." (D)

27b, 0: "I'll pass it up." (D)

30b, 0: "I'll pass it up." (D). The subject was asked how he would draw from 82 beads. He replied: "I would take 1. No, I would take 2, because that would throw me off the multiple of 3 again." (E?) From 67 beads he said he would take 1 "to avoid 66, a multiple of 3." (D). This subject did not advance

beyond Stage D in Series 1-2, although he seemed at the point of making the larger generalizations at several times.

Subject i

9b, 5: "I cannot beat you at 9. No matter how I draw you can take such a number as to reduce me to 6 which I can't win." (B)

11b, 0: "Now if I knew what my winning numbers were for 7, 8, 9, and 4, and 5, I think I could win."

" 4: "Did I win 6 before? I don't think so." (B)

" 6: "You reduce it to 6 no matter how I draw, and 6 is impossible." (B)

" 7: "The first party can't solve 11 if his opponent reduces the number to 6 (E). I have it. . . . If I can prevent you from reducing the number to 6, I can win."

12b, 3: "It reduces itself to this: Can I prevent you from reducing it to 6? You can leave 6 irrespective of what my move is. Twelve is composed of two 6's." (C)

13b, 1: "Now I want to prevent you from getting 6 or 12, so I must necessarily start with 2 (E?). (S draws 2 and E 1) Nine beads. There you have got me because I take 1 and you 2 and leave me 6, or I take 2 and you 1 and leave me 6." (C)

" 2: "I'll take the other chance and start with 1. No there is no use because I would be leaving you 12. Well, as far as I can see my only chance is to start with 2, but even then you can reduce me to 6 if I am not mistaken.

" 3: "It seems to me that I lose. If I take 1, you have 12 and win; if I take 2, you can bring it down to 9 and hence to 6 (C). If you can reduce me to 6 or 12 no matter how I draw, why I lose, that is certain. Let's see if I can beat you on 6. . . . No I can't beat you on 6, that is evident. I have an idea that it is multiples of 3 you win." (D) When asked how he would draw from 59 beads, he replies: "56 is the nearest multiple of 3 to 59. If you could reduce it to 57, you would have me. There again I can't move so as to prevent you. If I took 1, you would take 1 and leave me 57. If I took 2—Oh yes! If I moved 2 and followed it out consistently, I think I would have you beaten." (F, G?) The subject here stated that he had a glimpse of the right procedure at 11 beads, but that some distraction or other caused him to lose sight of it for a time.

Subject viii

11b, 20: "When there are 4 and it is E's draw, I can get it. My problem is to get it reduced to 4 when it is his draw. (E?)

12b, 18: "The idea strikes me that this cannot be gotten, and also that it is a multiple of 6 that cannot be beaten." (C)

13b, 18: "First I wanted to get 4 beads with my draw; I later found that I would have to work not from 4 but from 5. . . . Later I

changed it to 7. I thought I could work down to 4 from 7, and then I began to work from 9. I then found that I would have to begin to check off from the total; that is, if there are 13 beads, I draw 1 and then count 12. . . . *My object was to get 9 beads when it was my draw.*" (E?)

" 33: "I counted back to 11 and worked it out from 11. Here *I tried to get my opponent down to 6 instead of myself.* (E)

14b, 12: "I forgot that I was trying to keep from getting 6, and consequently for a good while I was trying to get 6, and not being able to win caused me a good deal of confusion. I have had this mixed up for some time."

15b, 7: "I am now working on 9 instead of 6." (F)

16b, 4: "I think that the point of the game is that the one who wins has to have the other one draw on some multiple of 3." (G)

The foregoing comments of subjects are merely milestones in their progress through Series 1-2, yet if the reader will refer to the position in the series where each comment belongs, he will appreciate the extreme slowness with which the simple meanings often developed and the lack of uniformity in their attachment to various elements of the series. It has already been shown that the lower critical numbers are usually wholly neglected for a time after their first discovery and that they are later re-discovered in about the same order as that of their first discovery. In general, each new acquisition of meaning—each advance to a higher stage in the foregoing classification—began with those critical numbers above the range of purely perceptual solutions, which had been most often repeated, and spread to the higher critical numbers much in the order of their frequency of occurrence in the work of the subject. Thus 6 was the first critical number, usually, to be re-discovered, and also the first to be definitely associated with a higher critical number. So also 6 and the immediately following critical numbers were usually the elements between which the multiple-of-three relationship was first apprehended though higher critical numbers were at the time known to the subject. Likewise 6 was usually the first critical number to advance to Stage E, and the higher critical numbers followed roughly in the order of their magnitude. It has already been shown that the reactions of any subject to a lower critical number were invariably more frequent than his reactions to a higher one. Thus in the elevation of critical num-

bers to higher levels of meaning frequency of repetition and nearness to a goal appear to be important factors, just as in the preceding section they were shown to be factors in the direction of attention to the critical numbers.

In the elevation of critical numbers to higher levels of meaning there was, however, considerable deviation from the order of their frequency of repetition. Thus 12 was often taken up into Stage C upon its first discovery long before 9 was rediscovered. This was clearly due to the simple multiple-relation existing between 12 and 6, by which the associations already built up about the latter were made to attach immediately to the former.[29] This spread of meaning through the agency of previously acquired associations might well be expected since the elements of our problems were not only familiar to the subjects, but were also closely related in past experience. But this calling up of familiar concepts by association is itself perhaps dependent in large measure on the frequency of their previous repetition in association with some element of the present situation. These irregularities in the process of learning did not greatly alter the general character of the learner's progress in the early stages of learning. The generalization of the formula for Series 1-2 into a suitable formula for all series of the first order[30] required from 1 to 6 further series for its completion; and the generalization of the latter formula so as to make it cover all continuous series[31] required from 2 to 10 additional series. It is therefore evident, notwithstanding frequent rapid advances by means of familiar concepts which were brought in by means of association, that the development of meaning was a rather slow and fluctuating affair, presenting some striking parallels to the formation of sensori-motor co-ordinations.

Another fact of interest, which is exemplified in some of the comments listed above, is the fleeting and unstable character of meanings in the early stages of their development. Individual critical numbers and generalizations based upon them were given

[29] For example, see the foregoing comments under Subject i, 12b, 3; Subject vii, 12b, 26; and Subject xiii, 12b, 36.

[30] Series in which the value of L is 1.

[31] Series in which all numbers between L and H may be drawn.

to slipping out of mind and to various distortions of meaning, which clearly reveal the weakness or utter lack of association with other elements of the situation. Thus Subject i, while working upon his seventh trial with 11 beads, appears to have had a fleeting insight into the significance of critical numbers as a means of actual control. Several other subjects spoke of a similar tendency for promising ideas to vanish during the attempt to formulate or apply them. Sometimes a critical number would be recognized and remembered well enough, but with an inversion of meaning somewhat comparable to the reversals of perspective in ambiguous drawings. Thus Subject xiv (11b, 29) said: "Whenever I win, I always have to leave 3 beads; that is, there must be 3 beads before I take my last draw." This sort of inversion of meaning often occurred shortly before the appearance of Stage E in the development of the concept of the critical number. After trying for some time at this point to get to draw from 6 beads subject viii said: "I forgot I was trying to keep from getting 6, and consequently for a good while I was trying to get 6." Another instance is found in the comments of Subject i quoted above (13b, 1). Similarly the idea of the significance of multiples of 3 often dawned and was lost, later to reappear, perhaps several times, before finally being applied in a thoroughgoing way with a full realization of its significance. Examples of the early instability of this idea are found in the comments listed above as follows: vii, 18b-22b; xii, 9b-30b, and xiii, 15b-27b. In tracing the further consolidation of the various elements of the concept of the critical number through succeeding series of problems, it is found that they are often thrown out of co-ordination by the appearance of some new element which requires adjustment and further generalization, or by the recall of some old hypothesis which has not been satisfactorily disposed of.

The interference of old and erroneous hypotheses with the formation and stability of correct generalizations, is worthy of further attention. The comments of Subject xiii, quoted above (27b, 2), show how the idea that odd and even numbers were possessed of special significance, came in to confuse him after

he had made a perfectly good generalization. This subject had played somewhat with the idea of odd and even numbers much earlier in the series, but during the greater portion of the work he seems entirely to have dismissed the idea, making no mention of it at all. The very common interference of erroneous hypotheses and irrelevant ideas which have once received attention without being definitely settled, will be further illustrated in the following paragraphs on trial and error in generalization.

3. *Random Hypotheses*

Having observed the slow growth of meanings and their instability in the early stages of development, we may turn briefly to the manner in which the elements of a situation were brought together and combined into adequate generalizations. It rarely happened in the early members of a group of series that the essential elements were directly sought out and held in mind without attempts at generalization until sufficient data were at hand to insure correct inference at once. This type of procedure was followed apparently by one of our subjects in Series 1-2, but was not carried out consistently in all later series. The work of all other subjects in the first series showed some loss or omission of essential data with consequent erroneous generalizations. These random hypotheses were often numerous and far afield. The rate of progress depended very much upon the thoroughness with which they were followed up and tested. Because of the unusual fullness of his comments the work of Subject xiv will be taken to illustrate the random efforts at generalization in the first series. The place in the series at which each comment occurred will be indicated in the manner already familiar to the reader.

9b, 3: "You can always win if you draw the same as I."
11b, 20: "If, with an odd number, you take an even number, I can always beat you if I work it right."
" 36: "Whenever I get a combination by which I can win, thereafter I must reverse when you reverse in order to win."
15b, 6: "I win some odds and some evens."
" 7: "I draw another conclusion, that my principle is not to follow you."

" 8: "Neither is it to draw opposite from you."

" 9: "Neither is it to alternate with you."

17b, 1: "I can always beat you on even numbers; I don't think I can beat you on any odds."

" 9: "There must always be 3 beads when you draw last."

18b, 2: "As long as you draw opposite to me I can't win 18."

21b, 7: "You win odd numbers by taking the opposite of my draw."

22b, 2: "The principle of alternation will not work with the even numbers."

25b, 1: "Even and odd numbers is not the criterion."

" 12: "There must be some relation between the total number of beads and the number of draws."

" 13: "When I win, the critical draw is the second or no later." For some time after this the subject's attention was absorbed in the number of beads he obtained in each trial.

31b, 12: "I haven't kept track of the numbers I won and those I lost."

32b 1: "To win an *odd* number of beads I draw an odd number." Here again the subject's attention is absorbed for some time in the number of draws and of beads obtained in each trial.

34b, 21: "2 from 34 leaves 32. Now if I draw 1 always and you 2, I would leave even numbers." (!)

36b, 1: "It seems that I can't win every third time."

37b, 4: "Of course I know you are going to take 2 all the time and I'll take 1, but I don't know how it will come out."

40b, 1: "I will beat you but I can't say how with any degree of certainty. I'll beat you twice and then you will beat me once. I notice that the numbers upon which you beat me are always divisible by 3." The subject is asked how he would draw from 59 beads and replies that he cannot tell. He then figures slowly but correctly to 59, after which he is asked how he would draw from 1004 beads. "I couldn't figure that up," he said, "I may win even numbers and you odd numbers. No, I won 37. It may be numbers that are divisible by 2. No, I don't get it. . . . It might be that you allow me to beat you on all numbers that are divisible by a certain number, but it isn't by 2 and it isn't by 3 or 4. . . . I have never figured out this drawing business; I do it wholly according to my feeling. But I watch the way you draw on your second draw and then take the opposite. . . . I am not sure what is the criterion, whether it is your first two draws or my first two."

40b, 5: "I ought to have noticed that 3-matter long before because I noticed so often that we drew 1 and 2."

41b, 4: "I knew I could beat; I had 3 beads after my second draw."

42b, 1: "I can't beat you on 42, it is divisible by 3. I know what you can beat me on but I don't know why."

43b, 4: "They go in pairs. I can beat you on 43 if I take 2 the first time (He loses). You haven't arbitrarily made up your mind not to let me beat you the first time, have you?"

" 8: "When I win I guess I must draw 2 the first time; but I can't win that way, because I've tried it and failed. . . . But if you happen to be drawing the opposite of what I draw the second time, I'll beat you. . . . I'll take 2 and then you'll let me beat you."

" 9: "Well, I thought all I had to do to beat you the second time was to reverse what I did the first. I see, evidently I have to start out with the same number in order to beat you on a given number."

" 10: "Three from 43 leaves 40. All right, I have it. I have an *even* number; I'll take 2."

" 12: "I had 15 draws—Oh yes, that is all right. I was thinking I could get out by leaving an *odd* number of beads."

44b, 1: The subject takes 2 and wins. "That confirms my rule," he says.

45b, 0: "I can't win 45."

46b, 1: "I can win 46. I'll have to start with 2. Yes, I must or I can't win, I believe."

" 3: S draws 1 and wins. "On the numbers on which I can beat you," he says, "I have to start out with 1."

47b, 1: "I can beat you on this. . . . Well, that rule didn't work (i.e., that S must always draw 1 first to win). Well, maybe the rule is that one time I draw 1 in order to win and the next time I draw 2 to win."

48b, 0: "Did I win 47 by drawing 1? No, I won it by taking 2. Then I can't win 48, of course."

49b, 1: "I'll draw here on the hypothesis that on the first one after the number I lose, I can win by taking 1, and on the second after it I must start with 2 to win."

" 2: "It worked. *Now this time I'll start with 2* (S loses). No, that isn't right."

50b, 1: "Now I'll start with 2 and beat you." Here the subject is taken back to the beginning of the series.

7b, 1: (S takes 2 and loses) "Now that's funny."

" 2: "I'll take 1 to start with."

" 3: S takes 2 and loses. "I've got to start out with 1 on 7," he observes, "and there is a lot of other numbers on which I have found that I must start with 1."

8b, 1: "I'll try 2 just to see. No, I'll try 1."

" 3: "Why isn't the rule that on the *even* numbers I've got to start with 2?"

" 4: "Yes, that's the rule."

9b, 1: "Oh, I can't beat you on 9, and *on 10 I've got to start with 2.*"

10b, 2: "Oh! I mustn't take 2 the first time; I must take 1."

11b, 2: "I believe my hypothesis was right about there being first the 1-draw and then the 2-draw. On 4 I had to draw—I forget just how I did draw on 4. On 10 I drew 1 and on 11 I took 2. I think I must take 2 on 11."

12b, 0: "I can't win 12; it is a multiple of 3."

15b, 0: "I can't get it."
16b, 0: "Now I'll take 1."
17b, 1: "I'll draw 2." The subject was then asked how he would draw from 59 beads. "Why," he said, "it goes this way. I draw 1 first, then you draw 2 and I keep on with 1. Then I beat you. The next time[32] I start with 2 and you start with 1, and I keep on with 2 and win. But I can't tell how a given number must be drawn on. I just know that it goes along 1 and 2. What is this number? (17) Well, on 17 I had to draw 2 didn't I? Well, I should guess that on 59 I should start with 2, and then you would draw 1 and I would continue with 2. (Why?) The only reason I say that is that it is just before 60, and 60 is a number that you beat me on just as you do on 18. Seventeen is just before 18 and 59 is just before 60." In reply to a question as to how he would draw from 43 beads the subject said: "I would draw 1 on 43, because on 42 I couldn't beat you, and with the numbers following those on which I can't beat you I must always start out on 1."

The random making of hypotheses upon the basis of momentary suggestions is here too evident to require further comment. Examples might be produced almost indefinitely from Series 1-2. A few examples from higher series and from the efforts of subjects to get a generalization of universal application for all series will perhaps be worth while.

The following evidences of random hypothesis-making are taken from the record of Subject viii. In attacking Series 2-3 this subject fluctuated a good deal among a number of hypotheses, as is shown in his comments:

7b, 2: "I think now that I will win on the basis of 7, i.e., I will try always to draw so that there will be 7 left to draw from."
8b, 2: "It appears that I can win on a basis of 5."
9b, 2: "It appears that 7 doesn't work."
" 3: "It doesn't work. I forgot about the 1."
" 6: "It will work on a basis of 6 as well as of 5, i.e., on multiples of both 3 and 2, I believe. But I am not certain."

[32] Note the possibility of confusion in the phrase, *"next time."* It may mean the next trial upon the same number or it may mean the next higher number. These two meanings fluctuate in the mind of the subject, owing perhaps to the ambiguous wording. The same ambiguous wording appears in the comments of this subject at 47b, 1 (p. 94), though here it is clear that the subject has the correct idea in mind. But it is the wording rather than the idea, that functions in the immediately following problems (47b, 1 to 7b, 3) in such a manner as to produce a number of errors. Several other instances were found of ambiguous wording which led other subjects into confusion and error.

15b, 3: "I am a little in doubt about the multiple of 3 and 2. It doesn't seem to work."

20b, 3: "Twenty can't be worked. I didn't think of that multiple until I had worked for a long time. It is a multiple of 5."

21b, 1: "Sometimes the multiple of 3 works and sometimes it doesn't. Either a multiple of 2 or a multiple of 3."

21b, 5: "At first I thought I could work on the basis of 2 or 3, but one of the multiples of 3 didn't come out right. I almost think now that I made a mistake and that I can't work on this basis."

26b, 2: "That hypothesis couldn't have been right because 12 is a multiple of both 2 and 3 (S won 12 repeatedly), so I have given that up."

The subject was tormented by these and numerous other hypotheses until he had solved all of the numbers of this series from 6 to 51 and then repeated the process up to 29 beads.

Perhaps the most striking exhibition of this sort of random effort was found in the attempts of subjects to formulate a general solution for all series. Here it was usually necessary to give them a list of the critical numbers of all discontinuous series in order to refresh their memories.[33] All of the subjects had made some discoveries, as for example, that all multiples of L + H are critical numbers. The generality of some of these earlier formulatonis was readily recognized if not already known. But some were not universal in their application, and here is where the effort began to be more sporadic. The subject would usually begin by restating one of his old generalizations or by formulating a new one on the basis of one or two problems, and then go from series to series testing it out until a series was discovered to which it would not apply. Upon finding a refractory series he would stop and attempt to modify his view to suit the case in hand and then pass on to test it out upon other series. Often the subject would have to modify his generalization for every group of series which were in any marked degree different from the preceding ones, sometimes discarding the notion only to return to it later in his fumbling efforts to find a generalization which would be applicable to all series. Some subjects failed after working for hours, and most of the others succeeded in getting a general solution only by a tedious "fitting on" of

[33] Series in which only L and H may be drawn. For a list of these similar to that presented to the subjects see page 7.

numerous hypotheses suggested by individual series and often wholly at variance with conditions which had been met repeatedly in other series. A better exhibition of random effort could not have been given by an animal in a problem box.

Nor is it to be supposed that in the application of generalizations to new situations we have an entire escape from trial and error procedure. Numerous errors and omissions of application point to the random character of efforts to apply generalizations. Certain hindrances to application and some sources of error are fairly apparent in the transition from the continuous to the discontinuous series of problems and in passing from the first to the second series of the latter group.

Three of the more important elements of their previous generalizations which subjects brought over into Series 1 or 3 are (1) the critical-number concept, (2) the idea of the serial relation of the critical numbers of the series, and (3) the knowledge of the fact that the common difference between successive terms of the various series of critical numbers is equal to L + H. No difficulty was experienced in applying (1) and (2) directly to Series 1 or 3. The trouble with the application of (3) arises from the fact that a new series of critical numbers is created by the restriction of the draws to L and H. The first term of this new series is 2 and the common difference is L + H as before. This new series of critical numbers should have given but little difficulty if the subjects had stuck to the old generalization for *primary* critical numbers and looked for an explanation for the *secondary*[34] critical numbers only. But there were too many pitfalls in the way. In the first place, the first critical number encountered was a secondary one. This refractory case immediately cast doubt upon the applicability of the old formula and lessened its chances of later consideration. Again, the secondary critical numbers fall into line with the primary ones in such a manner as to form a continuous series with a common difference of $\frac{L + H}{2}$. This is, of course, a coincidence peculiar to

[34] See above, p. 8 for definition of these terms.

this series and not common to all. Finally, the common difference of $\frac{L + H}{2}$ is 2, in this particular case, making all *even* numbers critical. Thus the mind of the subject is led unawares away from the general principle built up in previous series and directly into the rut of the old concepts of even and odd numbers. The power of these superficial relations and the familiar concepts aroused by them to divert the attention of subjects from the application of generalizations formed in previous series, is evident from the fact that twelve of the thirteen subjects who solved this series, stated their final formulae in terms of even numbers, though five of them had played with the old formula while working upon the early problems of the series.

Some interesting errors of application were made by Subject viii in passing from Series 1 or 3 to Series 1 or 5. After having generalized correctly for Series 1 or 3 he began his attack upon the following series correctly enough by saying: "I would think right away that 6 is the lowest critical number and that all multiples of 6 are critical." But he quickly got off the track and began to believe that all multiples of 3 were critical. Later he supposed that the critical numbers were all multiples of 9, but finally changed his conjecture to muitiples of 12. After obtaining a solution for this series he stated that the errors had arisen from his efforts to adapt the generalization of the preceding series—i.e., that "All multiples of 2 are critical"—to the present one. Examining the former series to find the basis of the distribution of critical numbers, he observed that 2 lies midway between 1 and 3, the L- and H-draws respectively, and straightway inferred that multiples of 3 were critical in Series 1 or 5 because 3 lies midway between the L- and the H-draw in this series. Finding this hypothesis to be incorrect he tried the sum of all the numbers between 1 and 5, i.e., 9, because it was observed that "2 constitutes all the numbers between 1 and 3. This failing, he "thought that a critical number had to be a multiple of all of the numbers between 1 and 5, i.e., a multiple of 12.

One of these errors was repeated by this subject in passing from Series 2 or 6 to Series 2 or 10. In generalizing upon the

former series he said: "Since about the beginning of the last series I have felt that it must go by multiples of 4, because 4 is intermediate between 2 and 6 and is a factor of both." Immediately afterward, when presented with Series 2 or 10, the subject said without hesitation: "I should say the critical numbers are multiples of 6 and numbers which are 1 above those multiples; that is, I add 2 to 10 making 12 which I know is a critical number. Then I reduce to 6—No, I didn't, did I?—as I had, in the other case, reduced 8 to 4."

In generalizing upon Series 3 or 9 Subject ii said: "The critical numbers are multiples of the difference between the high and the low draw." When presented with the next series, he said: "Immediately I take the difference between 4 and 12, which is 8. This formulation applies very well to all series in which H = 3L, but to no others. Although the subject did not explicitly say so, he undoubtedly carried this generalization over into Series 1 or 4 and Series 1 or 6 much to the detriment of his progress. While working upon 9 beads in Series 1 or 4 he decided that 3 was a critical number and said: "I feel that I am on the edge of something here . . . 3 I lose. It reduces to 3." Shortly afterwards he added: "I've been mistaken there. I lose 3 and 5. No, I win 3." After thinking this matter over for two minutes he continued: "I believe I *can* win 3, 6, and 9." A little later, after a short pause to determine how to draw from 9 beads, the subject said: "Now that's funny. I can *win* 9." The idea that multiples of 3 are critical numbers in this series seems finally to have been discarded at this point, but the generalization upon which it rested,—i.e., that the difference between successive critical numbers is equal to H — L,—persisted into the next series and seems to have been the cause of considerable difficulty there. The idea, carried over from Series 1 or 3, 1 or 5, and 1 or 7, that *even* numbers are critical, is also much in evidence in the records of Series 1 or 4 and 1 or 6. The ill effect of these generalizations from earlier series upon the progress of the subject through Series 1 or 4 and 1 or 6, is evident in his records. Though in the speed of progress throughout the entire experiment this subject ranks second, he is tenth in rank in Series 1

or 4 and 1 or 6 combined. The other nine of the ten subjects who solved all of the series devoted only 10 per cent of all their trials to these two series, whereas Subject ii devoted 32 per cent of all his trials in the entire experiment to the solution of them. It appears, therefore, that approximately two-thirds of the difficulty encountered by Subject ii in these two series, was due to the interference of certain generalizations carried over from preceding series.[35]

Many other examples of the difficulties attending the application of generalizations could be given but the principal types are perhaps sufficiently illustrated in the foregoing cases. The omissions and errors of application occurring in the records of all subjects are numerous enough to warrant the statement that, under the conditions of our experiment at least, the application of generalizations depends very much upon trial and error procedure. It appears that any incompleteness of analysis, either in the old situation from which the generalization is evolved or in the new situation to which it is to be applied, is likely to render application difficult and perhaps erroneous. There is no apparent reason why difficulties of this nature should arise from the conditions of our experiment more readily than from the practical situations of life.

4. *Summary*

Explicit generalization did not often occur until the number of beads presented was high enough to preclude the possibility of direct, perceptual foresight of the consequences of all possible draws, i.e., until the numbers presented were high enough to place a premium on the use of symbols in their solution. Num-

[35] It is of course possible that other factors contributed to this retardation of progress. Fatigue may have affected the progress in Series 1 or 4. This series was completed shortly before noon when the experiment had been in progress approximately three hours. The subject stated that he was becoming somewhat fatigued though not until after much difficulty had been encountered. However, the greater portion of the effort was expended upon Series 1 or 6 where fatigue conld not have been a factor. This series was begun after an intermission of one and a half hours, when the subject declared that he was thoroughly rested.

bers lower than this were seldom taken into account even in later attempts at generalization.

Several fairly distinct stages in the development of the critical-number concept were found. Each advance to a higher level of meaning began usually with that critical number, above the range of perceptual solution, which had been reacted to most frequently, and affected the higher critical numbers largely in the order of the frequency of their repetition. This is also the order of their numerical and temporal nearness to the goal, i.e., the end of the trial. In the early stages of their development the meanings involved in the concept of the critical number were found to be extremely unstable in character. The stability of these meanings gradually increased with continued reaction to the situations from which they were evolving.

The selection of the essential elements of a series and their combination into adequate generalizations was effected mainly by means of trial inferences, or random hypotheses, which were made upon the basis of only a few cases—often only one—and tested out more or less persistently before final acceptance or rejection by the subject.

In the application of generalizations to situations which are in part new, much difficulty was experienced, and a considerable amount of random modification often occurred before a successful adjustment to the requirements of the new situation could be made. False analogies arising from the observation of superficial relations often resulted in confusion and error. Sometimes the solution of new problems was much retarded by the attempted application of inadequate or irrelevant generalizations formulated from the elements of earlier series.

F. Transfer

The term, *transfer,* is used by Ruger broadly "to include the effect of any given experience on any subsequent one whether the effect results directly or by means of an idea, whether the transfer is one of method or of material, or of motor processes, and whether it is positive or negative."[36] The term will be used

[36] *Op. cit.,* p. 85.

in the same broad sense in this discussion. This usage is becoming fairly common. Ruger's defense of it need not detain us.

1. *Degree of Transfer*

Time and other limitations have not permitted such experimental evaluation of the various problems as would be required for a highly accurate measurement of the transfer of the effects of learning from problem to problem. However, the facilitating and, sometimes, the inhibiting effects of earlier upon later efforts, are great enough to give a fair insight into the causes and conditions of transfer without a very precise determination of its amount. A rough estimate of the relative difficulty of various problems and series of problems can be made from some of the experimental data and other facts.

Within a given series the higher problems are undoubtedly more difficult than the lower ones. If the subject's draws were determined by pure chance, two successive winnings would require 8 trials upon 5 beads, 32 trials upon 8 beads, 128 upon 11 beads, and 512 upon 14 beads. This increasing difficulty is sufficient to conceal the effects of transfer in many instances, especially in the lower numbers of the series. Though no experimental evaluation of the various problems of any series was attempted, the records of Group III when compared with those of Group IIA show that the solution of 14 beads was considerably more difficult than that of lower numbers of the series. Group IIA began with 4 beads and followed the same procedure as Group I. Group III followed the same procedure but began with 14 beads instead of 4. The number of trials required by each member of this group for the solution of 14 together with some data for comparison from Group IIA, are given in Table XII.

The subjects of Group III required a greater number of trials for the solution of 14 beads than were required by those of Group IIA for the solution of the first seven problems of the series, and over half as many as were required by the latter group for the solution of the first ten problems. Assuming that the two groups were possessed of equal ability to solve problems

TABLE XII

Group IIA	Total No. of trials required for the solution of all problems from 4 to 10 inclusive	Total No. of trials required for solution of all problems from 4 to 13 inclusive	No. of trials required for solution of 14 beads: Group IIA	Group III	
Subj. A	32	48	0	Subj. 1	8
" B	39	49	5	" 2	11
" C	25	68	2	" 3	15
" D	56	88	4	" 4	27
" E	34	106	4	" 5	29
" F	59	84	4	" 6	35
" G	36	112	22	" 7	42
" H	74	97	25	" 8	50
" I	47	159	2	" 9	60
" J	39	92	32	" 10	63
				" 11	86
				" 12	180
Total	441	903	100		606
Average	44.1	90.3	10.0		50.6

of this sort, the transfer from an average of 90.3 trials upon the first ten problems to the eleventh problem of the series was equal to 40.6 trials, or to 80.2 per cent of the average number of trials required for the solution of 14 beads when presented as the initial problem. Further profitable discussion of transfer from problem to problem within a series must await a careful determination of the difficulty of these problems.

From logical analysis it would seem that all series of the same order are of approximately equal difficulty and that series of higher orders are somewhat more difficult than those of lower orders.[37] Some experimental evidence of the substantial equality

[37] In Series 1-2 there are just four combinations of draws by which subjects can win non-critical numbers. Upon problems in which the initial number of beads is greater by 1 than a multiple of H + L the subject can win half of the trials by taking 1 bead at every draw. The other half he can win by taking 1 at his first draw and 2 at every draw thereafter throughout the trial. He can win one half of the trials upon problems in which the initial number is less by 1 than a multiple of H + L, by taking 2 beads at every draw. The other half of these problems he can win by taking 2 at his first draw and 1 thereafter throughout the trial. If L and H be substi-

of difficulty of Series 1-2 and Series 1-3, is found in the records of Groups IIA and IIB, which are summarized in Table XIII.[38] The subjects are listed at the left in the first column. The number of problems solved by each subject in each of the series and the number of trials and amount of time required for the solution of the series, are given in the succeeding columns.

tuted for 1 and 2, these four combinations of draws are effective for the solution of some problems in every series of the first order. But for all problems in which the initial number is removed from a multiple of H + L by an interval greater than 1, a new combination of draws must be made by varying the first draw. In Series 1-3 a new combination is required for only those problems in which the initial number of beads is an odd multiple of 2; i.e., for one-third of the non-critical numbers of the series. In Series 1-4 it is required for half and in Series 1-5 for three-fifths of the non-critical numbers, etc.

Moreover, the discovery of the successful combinations of draws is not favored at so early a point in the higher as in the lower series, owing to the magnitude of the H-draws which effect a reduction of small numbers to 0 before a sufficient number of repetitions has been made to impress the uniformity of response upon the mind of the subject. For the same reason the early discovery of the principle of drawing by opposites is less likely in the higher than in the lower series. Again, because of the longer interval between critical numbers in the higher series, a greater number of problems must be solved in these series than in lower ones in order to reveal a given number of critical numbers upon which to base and with which to test generalizations.

On the other hand, each critical number in the higher series is more firmly fixed in mind than those of the lower series by the more frequent reaction of subjects to it before the next higher critical number is presented. This would favor the more ready utilization of critical numbers in the higher series for purposes of generalization and so facilitate the solution of these series. This higher degree of learning is the only factor revealed by analysis which clearly favors the more rapid solution of higher than of lower series of the same order. How important these various factors are cannot be stated definitely without further experimentation. It does not seem unreasonable, however, on the basis of this analysis, to assume that the higher series are at least as difficult as lower series of the same order.

Critical numbers are relatively more numerous in higher than in lower orders of series, and their relations to one another are more complex. It is practically certain, therefore, that a series of a higher order is more difficult than one of a lower order.

[38] The subjects of these groups were members of a class in educational psychology, whose class work had been observed by the writer for from three to seven months. The division into groups was made upon the basis of reasoning ability as judged by the writer.

TABLE XIII

Group IIA

Subj.	Series 1-2 No. Problems	Series 1-2 Trials	Series 1-2 Time (in seconds)	Series 1-3 No. Problems	Series 1-3 Trials	Series 1-3 Time
A	10	48	1833	12	37	1130
B	15	62	1555	8	18	230
C	15	85	1840	7	13	248
D	16	112	4210	6	24	685
E	21	220	6450	7	15	130
F	74	300	19133	9	31	1545
G	27	332	7440	0	0	0
H	76	389	13663	8	16	312
I	81	617	15988	8	17	162
J	162	823	18572	88	524	10260
Total	497	2988	90684	153	695	14702
Average	49.7	298.8	9068.4	15.3	69.5	1470.2

Group IIB

Subj.	Series 1-3 No. Problems	Series 1-3 Trials	Series 1-3 Time (in seconds)	Series 1-2 No. Problems	Series 1-2 Trials	Series 1-2 Time
a	16	48	2545	9	14	300
b	16	50	2062	12	29	763
c	19	56	4545	9	15	407
d	16	69	2640	4	10	135
e	17	95	2090	3	7	70
f	23	179	9585	1	2	120
g	38	210	8455	6	14	270
h	26	337	10131	12	31	532
i	52	376	18648	3	6	150
j	98	776	20435	12	30	1054
Total	321	2196	81136	71	158	3801
Average	32.1	219.6	8113.6	7.1	15.8	380.1

The average number of trials per subject in Series 1-2 is 298.8, P.E., 51.9. In Series 1-3 the average number of trials per subject is 219.6, P.E., 46.3. The difference between these averages is 79.2, P.E., 69.5. In so far, therefore, as the calculation of unreliability means anything when based upon so few and so variable data as those at our command, these data support the view that there is no substantial difference in the difficulty of different series of the same order.

In terms of the number of trials required to solve the series, the transfer from Series 1-2 to Series 1-3 is 76.7 per cent. From Series 1-3 to Series 1-2 it is 92.8 per cent. The lower percentage of transfer in the former case is due almost wholly to the behavior of Subject J. The insight of this subject into the first series at the time of her successful generalization was unusually superficial and, contrary to custom, five days were permitted to pass between her completion of the first series and her attack upon the second. If we leave her record out of account, the degree of transfer for the group becomes 92.1 per cent. The transfer from either series to the other is here so great as completely to overshadow such differences as may exist in the difficulty of the two series.

The degree of transfer from series to series within the various orders of continuous series is shown in Table XIV. The percentages of transfer are based on the supposition that all series in any given order are equal in difficulty. The series are listed at the left of the table in the usual order. Individual subjects of the group are indicated by the Roman numerals at the top. Though not all of the series were specifically solved by all subjects the general solutions given for all series of an order were such as to indicate that all higher series of the order could be solved without manipulation of the beads. We have therefore figured the average number of trials per series just as if every series had been specifically solved by every subject.

The degree of transfer may be further traced from lower to higher orders of series. As already stated, the higher orders are probably more difficult than the lower ones, but in the absence of experimental data upon this point we shall assume that all orders are of equal difficulty and calculate the percentages of transfer upon that basis. The transfer from lower to higher orders, as found in the records of the thirteen subjects who solved all of the continuous series, is shown in Table XV.

Further evidence of transfer is found in the constant decrease, through successively higher orders, in the number of series required to be solved for the mastery of an entire order. The average numbers of series solved by actual manipulation of the

TABLE XIV

Transfer from Series to Series, Group I

Series	Subj. i	ii	iii	iv	v	vi	vii	viii	ix	x	xi	xii	xiii	xiv	Ave.	Per cent of trials saved as compared with first series of same order	Per cent of transfer from next preceding series
1-2	48	35	28	96	62	331	371	168	315	597	53	184	166	367	201.5		
1-3	7	23	10	35	15	30	23	56	16	45	34	11	20	1	23.3	88.4	88.4
1-4	14	0	0	27	3	4	8	30	3	3	0	16	35	0	10.2	94.9	56.2
1-5	0	0	0	10	0	0	0	2	2	0	0	0	18	0	2.3	98.8	77.4
1-6	0	0	0	0	0	0	0	0	9	0	0	0	16	0	1.8	99.1	21.7
1-7	0	0	0	0	0	0	0	0	0	0	0	0	19	0	1.4	99.3	22.2
1-8	0	0	0	0	0	0	0	0	0	0	0	0	2	0	.1	99.9	89.5
2-3	44	17	25	44	71	30	25	167	120	65	117	89	40	43	64.1		
2-4	0	4	84	0	29	1	13	71	54	26	21	45	8	115	33.6	47.6	47.6
2-5	0	0	60	0	0	0	0	20	15	2	21	1	0		9.2	85.6	72.3
2-6	0	0	0	0	0	0	0	2	0	0	14	0	0		1.2	98.1	86.9
3-4	0	0	0	0	0	1	12	41	10	40	20	3	16		11.0		
3-5	0	0	0	0	0	0	0	4	9	0	13	0	70		7.4	34.5	34.5
3-6	0	0	0	0	0	0	30	0	0	12	0	0	0		3.2	70.9	56.8
4-5	0	0	0	0	0	0	0	0	0	10	29	0	31		5.4		

TABLE XV

Transfer from lower to higher Orders of Series

Order	Ave. No. of trials per subject	Per cent of trials saved as compared with first order	Percentage of transfer from preceding order
First	230.8		
Second	103.5	55.3	55.3
Third	21.6	90.6	79.2
Fourth	5.4	97.7	75.1

beads in the various orders of continuous series were 3.38, 2.46, 1.08, and .23 for the first, second, third, and fourth orders respectively.

Owing to various irregularities which make their comparability uncertain, no attempt will be made to determine the degree of transfer in the discontinuous series. That much positive transfer occurred can readily be seen by referring to the data upon these series in Table III.

2. *Conditions of Transfer*

No attempt will be made to enumerate all of the conditions and sources of transfer but a few of the more important ones will be mentioned and discussed briefly.

a. Objectively Identical Elements.—In order to solve any of the higher problems of a series it is necessary to make all of the moves appropriate to the solution of all lower non-critical numbers of the series. All of the lower numbers of a series are therefore common elements of the higher problems of the series. Thus in Series 1-2 the numbers 4, 5, 6, 7, 8, etc. must all be solved in the process of solving the higher numbers of the series. That the solution of these lower numbers contributes to the mastery of higher ones is shown by the results reported in Table XII. Also the effect of the presence of common critical numbers is shown by the fact that subjects soon learned the necessity of avoiding certain critical numbers, and later of forcing the experimenter to draw from these numbers regardless of what might be the initial number of beads presented for solution.

Likewise in all series of a given order the lowest possible draw is a common element; e.g., in Series 1-2, 1-3, 1-4, etc. the low

draw is the same for all series. It is clear that this common element is responsible for a large portion of the transfer, for after the solution of Series 1-2, in which neither the high nor the low draw was usually recognized as especially related to the formula, it required only 41.7 trials on the average to generalize for all possible variations of the high draw with the low draw remaining constant. But it required 130.3 additional trials on the average to generalize for all possible variations of both the high and the low draw.[39] In other words, the mastery of an indeterminate number of series in which one of the two most essential elements is identical, required less than one-third as many trials as were necessary for the mastery of an indeterminate number of series in which this objectively identical element is lacking. And this notwithstanding a large amount of transfer through other elements from the former to the latter group of series. Other common elements of this sort might be mentioned, as for example, the different sequences of E's draws (described above, p. 18), but with further enumeration it becomes increasingly more difficult to distinguish between subjective and objective identities.

It must not be supposed that these objectively identical elements were invariably recognized at once as common to the various problems of the series. Even the most obvious of them were often very slow in impressing themselves upon the minds of some subjects. But the absence of explicit awareness of these elements is no warrant for the contention that no transfer was mediated by them in the early stages of the experiment. The influence of unrecognized factors of some sort is clearly evidenced by the fact that subjects were frequently able to solve problems though unable to say how it was done or to give better reasons for correct draws than that they "seemed right."

b. Subjectively Identical Elements.—Many of the more important elements which came to be regarded as common to a number of problems are not, however, present in the same objective sense as those mentioned above. Such identical ele-

[39] The intermediate draws may be neglected since they varied in the same manner in both cases and generally attracted little or no attention.

ments are, in large measure, the result of the generalizing activities of the subject. Judd has insisted at length upon the importance of this type of common element as a source of transfer.[40] Elements of this sort are fairly numerous and constitute the principal medium of transfer as found in this study. The high draw, or H, for example, is clearly dependent upon generalization for its status as a common element in all series, as is also the low draw, or L. These two symbols with their associated meanings are undoubtedly the most effective instruments of transfer for all subjects who mastered any considerable number of series. It is through the medium of these two minor concepts, largely, that the meanings developed in and about specific critical numbers, become detached and generalized and finally applied to other appropriate elements in the same and other series. On the basis of these and other generalized elements every subject, who had the perseverance to stay with the task, sooner or later acquired the ability to solve at sight series in which neither the low nor the high draw was common to the preceding series in a purely objective sense.

Common elements of these two types were not always easy to distinguish, however. Specific elements of the objective sort were often apprehended in general terms at their first presentation, or, at any rate, the subjects' verbal reactions to them were couched in general terms. Thus the fact that, after reducing the number of beads to a multiple of L + H, E always drew 1 when S drew 2 and 2 when S drew 1, was often noticed and alluded to by the subject as "drawing opposites."

c. Generalized Methods of Procedure.—Closely related to the generalized elements of content and depending largely upon them were certain generalized methods of procedure. The most common of these types of procedure to develop in the first series and persist through later series, consisted of some sort of systematic alternation of various sequences of draws designed to prevent useless repetition of unsuccessful variations or to exhaust the possibilities of variation. One subject described his procedure as follows: "First I take 1 and then the same as you took (for

[40] Judd, Psychology of High School Subjects, p. 414.

the remaining draws of the trial), then 1 and the opposite; then I take 2 and the same, and finally 2 and the opposite." The first and third of these sequences of draws were not successful. The others were successful approximately half of the time upon non-critical numbers.

There is good evidence that the erroneous sequences were brought in by a mere contrast association between the words *opposite* and *same*. Here are the circumstances: After winning twice in succession from 14 beads, the subject said: "If I take 1 when you take 2 and 2 when you take 1, I win; i.e., if I take the *opposite* of your draw." After three trials upon 15 beads he said: "The principle I discovered seems to hold now for you." He then began immediately to follow the experimenter's draws, i.e., to draw the *same*. This failing in the first three trials, he changed again to the opposite in the fourth trial, and added at its completion: "When that was reduced to 12, I tried taking the opposite of what you took, but it didn't work." He then began to follow the procedure described above and continued for over an hour to the very obvious detriment of his progress. Finally the erroneous sequences were eliminated and his progress somewhat accelerated. This modified procedure was carried forward to later series.

Another helpful procedure which was used quite effectively by some of the subjects was to begin the search for critical numbers with 0 which was regarded as the first critical number in every series. This had the effect of limiting the number of problems to be solved in locating a sufficient number of critical points for generalization. The procedure was adopted by Subject vi at the beginning of Series 1 or 6 and by Subject ii near the beginning of the experiment.

An effective method of limiting the scope of attention in the discontinuous series without sacrificing any useful variations, was to subtract the highest multiple of L + H contained in the number presented for solution, and work upon the remainder which was thus brought well within the range of perceptual control. In Series 2 or 5, for example, if 17 beads were presented, the subject would find the difference between 14 and 17 and deal

directly with this small remainder. After the first draw had been determined in this manner it was usually possible to make the remaining draws mechanically upon the principle of drawing opposites. This procedure was followed more or less persistently with good effect by about a third of the subjects.

A less definite though perhaps more profitable form of systematization consisted in a better balancing and correlation of rational thinking with overt trial-and-error procedure as the experiment progressed. In the early problems much of the manipulation was extremely ill-directed, or undirected, and much time was wasted in fruitless attempts to generalize upon insufficient data. The later work of most subjects showed much improvement both in the purposeful direction of trial-and-error procedure and in the matter of judgment as to when enough data were in to warrant generalization. This process of empirically working out a proper balance and correlation between reflective thinking and overt trial and error, is probably one of the main sources of progress in all successful attempts to deal at all extensively with novel data.

d. Effect of Thoroughness of Learning upon Transfer.—It might be expected that the superior ability which enabled one subject to outrank another in the mastery of Series 1-2 would give him a proportionate advantage in the solution of the later series. As a matter of fact, however, subjects whose speed of progress in the solution of Series 1-2 is above the median show no superiority in the immediately following series over those whose speed is below median in the first series. This is true of all groups of subjects who solved as much as two successive series of problems, as is shown in the following data. The rate of progress is expressed in terms of the number of trials required for the solution of a series:

Although the subjects whose rate of progress was below median required over five times as many trials for the solution of the first series as was required by those whose rate was above median, they were able to solve the second series in practically the same number of trials as was required by their presumably more gifted fellow subjects. The degree of transfer from the

	Group I (14 subjects)		Group IIA[41] (7 subjects)		Group IIB (10 subjects)	
	First series	Second series	First series	Second series	First series	Second series
Total trials of subjects above median[42]	488	144	307	92	318	75
Total trials of subjects below median	2333	182	1638	64	1878	83

	First series	Second series
Total trials required by 16 subjects above median,	1113	311
Total trials required by 16 subjects below median,	5849	329

[41] The record of Subject J is here omitted because of an irregularity of procedure mentioned above on page 110.

[42] Above median here means above median speed of progress as in the foregoing text.

first to the second series, in terms of the number of trials required, was on the average 94 per cent for the subjects whose rate of progress in the first series was below median and only 72 per cent for those whose rate was above median. That these group averages do not grossly misrepresent the individual records is evident from the fact that only 4 of the 16 subjects whose rate of progress was below median in their respective groups in the first series, were surpassed in the percentage of transfer by any of the 16 subjects whose rate was above median. Correlations by the rank method between the quickness of mastery of the first series, as measured by the number of trials required, and the degree of transfer from the first to the second series, give coefficients of —.803, —.914, and —.819 for Groups I, IIA, and IIB respectively.

The thorough acquaintance with the elements of the problems, which is acquired by the initially slow subjects during their long-continued efforts to find a solution, is undoubtedly the main cause of the relatively high degree of transfer found at this stage of the work of these subjects. The records show clearly that the initially slow subjects generally surpassed the more speedy ones in the number and variety of elements which were discovered in the first series. From their four-fold more numerous responses to the same objective situations, it is not unreasonable to suppose that many of the associations formed by the slower subjects

were more firmly fixed than corresponding associations formed by the subjects whose progress was more rapid. The difference between the subjects whose initial progress was slow and those who progressed rapidly at the start, is not so much a matter of difference in the speed of formation of new associations as in the power to utilize these associations for a definite purpose. But the ready utilization of associations in the early stages of their development results in the accomplishment of the end before a sufficient number of repetitions has been made to fix the associations thoroughly. It also limits the number and variety of associations formed. Therefore the points of contact are usually neither so intimate nor so numerous, and transfer at this point is generally not so great, when the initial progress is rapid as when it is slow.

3. *Negative Transfer*

Numerous facilitating and inhibiting factors enter in at every stage of the work to determine the rate of progress of subjects. On the whole, the facilitating influences so far outweigh the inhibiting ones as generally to make it difficult to detect the presence of the latter, to say nothing of the accurate measurement of their effects. Occasionally, however, the presence of these negative factors becomes clearly evident from the comments of subjects, and sometimes their effects become so obvious as to make a rough quantitative statement of their strength possible. Thus it has been shown above (pp. 80-81) that the progress of Subject ii through Series 1 or 4 and 1 or 6 was only approximately one-third as rapid as might reasonably have been expected but for the interference of certain superficial associations which were formed in the process of solving a few of the preceding series. Two clear cases of negative transfer were also given above (pp. 79-80) from the record of Subject viii.

A very similar case of negative transfer from Series 1-2 to Series 1-3 is found in the record of Subject ii. In his generalization upon the data of the former series he said: "I can't win any multiple of 3." When presented with the first problem on Series 1-3 he said at once: "I suspect right away that it will be

a multiple of 5 here—3 plus 2 equals 5." After his first trial, however, he said: "I was mistaken. I see now that 3 plus 2 plus 1 equals 6. Evidently I *can* win multiples of 5. It must be multiples of 6." This idea was wrongly confirmed by his first trial upon 6 which was a failure. His pleasure at the seeming confirmation of his theory was evident from the tone of his remarks. "By Jove," he said, "you won that!" The persistence of this hypothesis is shown in his later attempt to account for his inability to win 8 beads. "If I take 1," he said, "you might take 1 and leave 6, and I can't win 6, so I can't win 8." The persistence of this notion is again apparent in his final attempt to generalize at the conclusion of the series: "I add the numbers which one is permitted to take and find that the multiples of the resulting sum cannot be won." This significant slip of the tongue was quickly corrected, however, as follows: "No! No! I add the last number to the first because that would give you the total number which could be taken with two draws."

If the first two series are equal in difficulty, the algebraic sum of all positive and negative transfer effects was only 31.4 per cent for this subject, though the average for all other subjects amounted to 89.1 per cent, and for the other six above-median subjects of Group I, 73.3 per cent. If the ratio of the number of draws required for the solution of the second series to those required for the first series had been the same as for all other subjects combined, he would have solved Series 1-2 in 4 trials. If this ratio had been the same for him as for the other six above-median subjects of Group I, he would have solved the series in 10 trials. Thus, figuring on a conservative basis, the negative transfer occasioned by the false analogy between the two series, and perhaps by some other inhibiting factors, was equal to 13 trials, or more than one-third the number required for the solution of the first series, and more than half the number required for the second series.

At the conclusion of Series 1-3 Subject iii generalized for all series as follows: "In general, you have got to add 1 to the highest number you may draw, and multiples of that sum cannot be won, e.g., for Series 1-3 it would be multiples of 4, for Series

1-4 it would be 5." Later in generalizing for Series 2-3 he said: "Combinations of 5 *and 1 above* are losers, i.e., numbers ending in 1 or 6." In Series 2-4 this subject did very well at the beginning. At 10 beads he said: "I win by leaving 6." Later when working with 11 beads he reiterated in more general terms: "When the starter can leave 6 beads, he wins it." After two trials upon 12 beads he added: "Combinations of 6, 12, etc., if they can be left by the one who draws last, win; i.e., the player who can leave 6, 12, etc. wins. After the second trial upon 13 beads the subject generalized as follows: "Combinations of *1 above any number*[43] or *1 above* any combination lose." He was asked to give an example and replied: "Well, 4 plus 1 equals 5. Combinations of 5, 10, 15, etc.; or combinations of 6, as, 6, 12,—no, combinations of 5 and 1 above, as, 6, 11, 16, etc. . . . anything that you can take away enough from to leave 5 you win."

Note how neatly the idea, "Add 1 to the highest draw," slips in from the generalization for all series of the first order to vitiate his reasoning while he is attempting to apply the "one-above" element from the formula for Series 2-3. Such a confusion could never have occurred if in the first formula "Add 1 to the highest draw" had been generalized into the more widely applicable form, "Add L to the highest draw"; that is to say, if the 1 to be added to the highest draw had been properly associated with the lowest possible draw in each series. Such association would certainly have prevented the addition of 1 to 4 in Series 2-4 to find the number whose multiples are critical numbers. The erroneous application of this idea is perhaps due in part to the similarity of its wording to that of the "one-above" element of the formulation for Series 2-3, for it was during the attempt to apply the latter element to Series 2-4 that the former one insinuated itself into the process. In the last instance, it will be observed, the wording of the two ideas is exactly the same, i.e., "*1 above* any number" and "*1 above* any combination."

[43] Probably meaning 1 above the high draw in any series, since he had already generalized the high draw but had said nothing concerning the low draw.

The subject's illustration shows clearly that this ambiguous wording referred to the two diverse elements mentioned above.

A somewhat similar case of negative transfer occurs in the transition from Series 3-4 to Series 3-5 in the record of Subject xiii. The reader will recall that the groups of primary critical numbers are always composed of L critical numbers each, and that the interval between such groups of critical numbers is always equal to H. When, therefore, a new series is formed by adding 1 to the high draw of the old series, the interval between successive groups of primary critical numbers is increased by 1. The groups of critical numbers, however, remain unchanged in size. But if a new series is formed by adding 1 to the low draw of the old series, the magnitude of the groups of critical numbers is increased by 1 and the interval between successive groups remains unchanged. The subject's error at this point consisted in applying the latter principle instead of the former. At the conclusion of the preceding series he had said: "The critical numbers are multiples of 7 and 1 and 2 beyond these multiples." At the presentation of Series 3-5 he said immediately: "This will be multiples of 7 and 1, 2 and 3 beyond." This attempt to apply the wrong principle is the only plausible explanation of the negative transfer which occurred at this point. There were no interruptions and no apparent fatigue. Such incorrect choice among principles which are but vaguely understood and insecurely held in mind recalls the difficulty encountered in the attempts to generalize upon the elements of a series in the early and unstable stages of their abstraction. These facts emphasize the importance of thoroughness in learning for positive transfer through generalization of experiences, and point to the danger of negative transfer from vagueness and under-learning.

4. *Summary*

The ability of subjects to solve the later problems of the experiment was much modified by the effects of learning in the mastery of earlier problems. Both facilitating and inhibiting factors were operative, though, on the whole, the former so far predominated as to leave a large balance of positive transfer.

This is true whether successive problems in a series, successive series of the same order, or successively higher orders of series are taken as a basis of comparison.

So far as our analysis goes, the greater the objective similarity between successive problems or series of problems, the higher is the degree of transfer. But it is not always possible to distinguish between objectively identical elements and common elements which owe their identity to the generalizing activities of the subject in apprehending them. It is, however, clear that a very large portion of the transfer observed, was effected through the medium of common elements of the latter sort, i.e., through concepts, and general principles, methods, attitudes, etc.

Some difficulty was encountered in the later series in attempts to apply generalizations from preceding ones. Misapplication of such generalizations often resulted in considerable loss of time and effort. False analogies, which were found to be the principal source of erroneous applications, resulted in the main from incompleteness of analysis either of the old problems upon which the generalizations were based, or of the new problems to which they were to be applied. Another source of negative transfer was found in the occasional ambiguous wording of generalizations, resulting in either their distortion or misapprehension.

There was, on the whole, much more transfer from the first to the second series in the work of subjects whose initial speed of progress was slow than in that of subjects who were more speedy at the start. This, apparently, was due in large measure to differences in the thoroughness of the acquaintance acquired with the elements of the first series.

G. Effect of Age and Education

Though the data at hand do not warrant a separate discussion of the effects of age and education on each of the processes of abstraction, generalization, and transfer, it may be worth while briefly to present such data as we have regarding the influence of these factors upon the general ability to solve our problem. Only two of our subjects differed enough from the general level of the group to warrant any attempt at comparison upon this point.

One of these subjects was in the preliminary group. The other is Subject iii of our major group of subjects.[44] The record of the latter will be mentioned first.

Subject iii of Group I was a thirteen-year-old boy who was in about the middle of his first year in high school. "I believe he is the brightest boy I have ever known," was the reply of his teacher in mathematics when asked concerning the mental ability of the boy. The father of this boy has shown remarkable genius in the business world. A comparison of the record of this subject with those of nine other subjects of Group I can be made by turning back to Table III. It will be noticed that Subject iii solved all the problems of the entire experiment in 8,425 seconds—considerably less time than was required by any other subject. Subject i, the next speediest in point of time, required 11,215 seconds, or 34.3 per cent more time than was required by the younger competitor. However, Subject i required only 165 trials whereas Subject iii required 388. Subject ii also finished with fewer trials than were required by Subject iii. Again, fewer individual problems were solved by each of these adult subjects than by Subject iii before a satisfactory generalization for all series could be formulated. Thus the lead in time which Subject iii gained over Subjects i and ii was lost in the greater number of reactions required for the attainment of a given degree of mastery of the problems. Subject iii appeared to be less conscious of method, somewhat more reliant upon trial-and-error procedure, and quicker in reaction than Subjects i and ii; but in no respect did his responses vary so much from those of these two adults as did those of some of the other adult subjects.

The other boy who participated in the experiment was eleven years old and had just completed the work of the seventh grade in the elementary schools. This subject served in the preliminary experiments in which fewer of the discontinuous series were given. His record upon these series is not, therefore, strictly comparable with those of any of the later subjects. Fortunately

[44] With the exception of one senior college student all of the adult subjects of Group I were graduate students or instructors at the University of Chicago.

the father of this boy, a doctor of philosophy, solved the same problems as the son in the same order. Hereafter we shall refer to the son as K and to the father as J. The only difference in the procedure with these two subjects was that J did the work in two sittings on successive days whereas K worked five successive days but with much shorter periods. It does not seem probable, however, that the relative speed of progress of the two subjects was seriously affected by this difference in the length of periods of work, since J stopped work in each instance as soon as he began to feel fatigued. The records of these two subjects are given in Table XVI.

TABLE XVI

Summary of Records of Subjects K and J.

	Subject K		Subject J	
Series	No. Trials	No. Problems Solved	No. Trials	No. Problems Solved
1-2	34	9	34	17
1-3	13	6	18	6
1-4	1	1	0	0
1-5	0	0	0	0
2-3	4	2	17	10
2-4	3	2	—[45]	—
2-5	8	2	4	2
3-4	2	2	10	8
3-5	3	1	0	0
3-6	0	0	0	0
5-6	1	1	0	0
3 or 1	6	4	5	5
2 or 6	22	16	8	8
3 or 9	13	9	6	9
2 or 7	36	21	14	16
Total	146	76	116	81

[45] Accidentally omitted.

There is no marked difference in the work of these two subjects. They required the same number of trials for the first series, after which K took the lead throughout all of the remaining continuous series but was surpassed by J in the discontinuous series. However, the lead established here by J was so small that it might easily have been lost if the work of the two subjects had continued further along parallel lines.

It is worthy of note that this eleven-year-old boy solved all of the continuous series in 69 trials though the best record established by any member of group I, that of Subject ii, was 79 trials. The differences in procedure in the work of K and that of Subject ii were in the length of work periods and in the fact that K (as also all other subjects who served in the preliminary experiments) did not have the stop-watch, dictaphone, and metronome before him and he worked with matches instead of beads. The effect of these differences upon the rate of progress was probably not great.

Considering the complexity of the mental processes involved and the limited amount of data at hand, it would be rash to attempt an explanation of the essential equality in the achievements of these children with those of the most capable members of our highly selected group of educated adult subjects. Further experiments are now being undertaken upon comparable groups of children and adults with a view to the analysis and comparison of their reactions to various sorts of novel problems.

IV. DISCUSSION AND CONCLUSIONS

For convenience of treatment we have divided the learning process here under study into three stages: (1) the abstraction of elements from the problematic situation, (2) the combination of essential elements into higher conceptual units, and (3) the application of these higher unitary processes to situations other than those out of which they arose. Some elements advance into the later stages before others emerge into the first, and some telescoping of successive stages occurs with many of the elements, so that there is considerable overlapping of stages in the process of solution as a whole. Yet these stages are characteristic enough of the growing adjustment to individual elements and to the problem as a whole to warrant their separate treatment.

Each stage begins with diffusion and multiple response, and progresses through elimination and the preservation and gradual co-ordination of essential responses. Trial and error is the method of procedure in all stages, but the materials among which variations occur are somewhat different and the field of variation is progressively narrowed in the later stages.

A. *Abstraction of Elements.*—Probably none of the elements of the problems were wholly new to any of the subjects but the essential elements were sufficiently mingled and fused with unessential ones to require considerable analytic activity before the solution of a series was possible. The prevalence of random manipulation in the first stage and the notable failure of all attempts at anticipatory analysis, are in accord with Ruger's observations upon the close external similiarities existing between human and animal methods of learning.[1] Our data enable us, in a modest measure, to carry the comparison beyond the superficial resemblances into the selective factors which determine the course of learning. This is not an easy task, however, owing to the diversity of opinion among experimental psychologists con-

[1] *Op. cit.*, p. 12.

cerning the selective factors in animal learning, and to the necessarily tentative nature of inferences as to the factors involved in so limited a study as our own.[2]

Perhaps the most important factor in the abstraction and selection of the essential elements of our problems, is frequency of repetition of responses to those elements. It has been shown (see above, pp. 40-47) that attention quickly came to be largely dominated by those elements of the problematic situation which were most often repeated. Likewise those elements which served most frequently as stimuli were the first to acquire definite meanings and special value for purposes of control.

Usually the most frequent objective element of response was the subject's notation of the number of beads remaining before each of his own draws. Though this type of counting did not occur with all subjects—at times some counted the numbers already drawn out by either the subject or the experimenter or by both, or seemingly did not count at all—yet it appeared to be the dominant form of response of the rapid learners, and indeed of all learners during periods of rapid progress. This counting reaction to the various numbers presented was clearly a basic element in the formation of the associations upon which the critical-number meanings were founded. It cannot be maintained that the frequent repetition of this important element of response was merely an effect rather than one of the causes of learning, since the frequency of its occurrence tended rather to be diminished than to be increased by learning.[3]

Some doubt has been raised as to the possibility of change in

[2] See Watson, "Behavior," pp. 267-268, and Joseph Peterson, "The Effect of Length of Blind Alleys on Maze Learning," Behav. Mon., 1917, 3, No. 4.

[3] Speaking of the status of frequency and recency as factors in maze learning Joseph Peterson says: "It cannot be too strongly pointed out . . . that this increasing percentage of reactions agreeing with the expectations based on recency and frequency effects, as learning advances from the first random stages toward the establishment of a regular habit, cannot be safely regarded as evidence that learning is brought about by recency and frequency factors: our evidence seems to justify the contrary conclusion, that this increase in reactions favoring recency and frequency factors is the *result* of the learning." (Peterson, Joseph. "Frequency and Recency Factors in Maze Learning by White Rats," p. 359.)

the order of responses in maze learning through the operation of frequency factors.[4] Even if the argument is well-founded, it is difficult to see how it could affect the explanation of learning in the present study, since both the direction of attention towards the critical numbers and the growth of the critical-number meanings may well be regarded as the effect of the mere formation and strengthening of associations between simultaneous or successive elements of stimulation and response. There is no necessity for change in the order of functioning of these associations.

It need hardly be said that we do not mean to contend that analysis proceeds upon the basis of the principle of frequency alone. As previously pointed out, the order in which the various critical numbers were abstracted is almost the exact order of their nearness to the goal, or end of the trial. This is in accord with the observations upon the maze-learning of rats. Commenting upon Hubbert's experiments, upon the basis of which backward elimination had been denied, Carr says:

> Determining the average number of trials necessary to eliminate each cul de sac, the order of elimination for the entire group was 6-5-3-4-2-1, where the successive errors are numbered in order from the entrance. This order gives by the rank method a positive correlation of .943 between quickness of elimination and propinquity to the food box. The average number of trials necessary to eliminate the last three errors was less than that for the first three for 90 per cent of the rats. Surely there is a very pronounced tendency for the errors to be mastered in proportion to their nearness to the food box, and the deviation from the *exact* correlation for each rat may well be due to the operation of other causal agencies.[5]

In summarizing her data concerning the order of elimination of errors, Vincent says:

> The elimination of the final members of the series first is not only true of the groups as a whole but also of the individual animals.[6]

[4] *Ibid.*, pp. 347-348.

[5] Carr, H. A. "The Distribution and Elimination of Errors in the Maze," Jour. Animal Behav., 1917, vol. 7, p. 146.

[6] Vincent, Stella B. "The White Rat and the Maze Problem," *Ibid.*, 1915, vol. 5, p. 371. Regarding the applicability of Vincent's data Hubbert and Lashley say: "In the form in which this is presented, however, Vincent is

Peterson finds the same general backward elimination of errors,[7] and even Hubbert and Lashley agree that

> when the averages of very large groups of animals are taken, there does seem to be a progressive elimination of errors from the food compartment to the entrance of the maze.[8]

What are the factors upon which the regressive order of mastery of the elements of the maze and of our own problems, is based? Though some of the suggested agencies in the determination of the order of learning in the maze are obviously inapplicable to our problems, it is not improbable that the determining factors are in part identical in the two forms of learning. It may be said at once that we do not assume with Watson, that the demonstration of a backward order of mastery need give any comfort to the advocates of the retroactive effects of *pleasure* as a causal factor in learning. As a matter of fact, the goal of most trials of our subjects was defeat; and here, as elsewhere, defeat usually gave rise to considerable evidence of disappointment or annoyance.

It may be argued that no explanation is necessary, beyond the greater frequency of response to lower than to higher critical numbers. This argument would be in essential agreement with Peterson's explanation of the order of mastery of the elements of the maze.[9] That other factors entered into the determination of the order, however, is clearly evident from the records of some of the subjects of Group III. While wrestling with 14 beads as their initial problem in Series 1-2, some of these subjects made nearly all of their errors in drawing from 14, or 11 beads; and, in consequence, reacted practically as often to 9 as to 6 beads. Yet 6 was recognized as a critical number some time before 9 was so regarded. Further evidence that the order of

not justified in applying the data to the problem. She makes no statement as to what constituted a trial in her experiments and in two 'typical records' . . . we find that the rat, after reaching the food, was allowed to return and re-explore the maze." (*Ibid.*, 1917, vol. 7, p. 132.)

[7] *Op. cit.*, pp. 360-361.

[8] Hubbert and Lashley, "The Elimination of Errors in the Maze," Jour. Animal Behav., 1917, vol. 7, No. 2.

[9] Peterson, Joseph, *op. cit.*, pp. 338-346.

learning was affected by the relative nearness of the elements to the goal, is found in the records of two subjects for whom the procedure was so modified as to eliminate the frequency factor, except in so far as this factor is itself determined by proximity to the goal.[10] All of the critical numbers learned by these two subjects were learned in the exact order of their nearness to the goal.

Attempts to solve the higher numbers by merely hitting upon a correct sequence of draws by a sort of long-range trial and error, had played a minor rôle in learning in the former experiments, but such attempts were practically useless here. Progress in analysis, which had previously been confined to the elements near the goal, was here almost entirely so confined.

At the beginning of their work our subjects (and this is true of all the groups) were able to foresee the outcome of a trial from a distance of not more than the first or, rarely, the second critical number above the goal, and that only with some doubt and uncertainty. But with continued repetitions the foresight of consequences became clearer and more certain until finally the critical number next above the goal became so closely associated

[10] The first of these subjects, a boy of fourteen years of age, worked upon 30 beads in Series 1-3 as his initial problem. To avoid the suggestive sequence of critical numbers in exact multiples of 4, success in a trial was defined as *forcing one's opponent to draw last.* The critical numbers then become 1, 5, 9, 13, etc. Again, the simple, suggestive sequences of the experimenter's draws, followed in the earlier experiments, were abandoned in favor of more difficult sequences which would tend to resist formulation and so impel the subject to attend more exclusively to the number remaining before each draw.

This problem was found to be too difficult. After 73 trials requiring upwards of two hours of time the subject had discovered only *5, 9,* and *13* of the critical numbers above 1, and had made practically no progress in the organization of the few facts observed.

A simpler problem, consisting of a similar variation of Series 1-2 with 25 as the initial number, was therefore given to the second subject. This subject succeeded, after 72 trials occupying a little less than two hours, in discovering all the critical numbers in his problem and in formulating his discoveries into a suitable generalization for the entire series. The number of draws from the various critical numbers was here almost exactly uniform, and all of the critical numbers were discovered in the exact order of their magnitude.

with defeat as to constitute a new point of orientation from which to push the analysis back one step farther. Thus, after its discovery and a sufficient number of repetitions to associate it firmly with the next lower critical number, each critical number became in effect a new goal. Two important consequences follow from this fact: (a) the next higher critical number was brought within range of direct and more or less clear apprehension of the consequences of each of the possible reactions to it, and (b) the time intervening between the occurrence of each of the higher critical numbers and the full realization of defeat, was reduced by a number of seconds.

It may be argued that (a) would result merely in the arousal of a new type of reaction to the critical number next above the new point of orientation, which, by frequent repetition, would gradually get itself established as the invariable response. This fact is freely admitted. But since the arousal of this sort of response is directly dependent upon the proximity of its stimulus to the goal, its repetition can hardly be used as evidence that frequency factors are alone responsible for the regressive order of mastery of the elements of the problem. On the contrary, the retroactive influence of the goal in selecting and consolidating the closely preceding essential reactions is here clearly evident.

Whatever causal agencies may have been involved, there can be no doubt that the rate of formation of associations between the goal response (the positive recognition of defeat in the current trial) and the various critical numbers varied directly with the relative nearness of these numbers to the goal. There seems to be no good reason why association should not be facilitated by a close temporal approximation of the various elements to be associated. That such facilitation does occur is suggested by certain experimental observations by Carr and Froeberg upon animal and human learning respectively. Carr says:

> The final co-ordination consists of an association between each act and the sensory aspects of the preceding act as well as a distinctive motor attitude resulting from the same. The relative efficiency of the two stimuli in determining the choice varies with the individual. The problem was mastered quickest by

those animals that relied mainly upon the factor of motor attitudes in making their choice. This fact suggests the hypothesis that the speed of learning is to some extent a function of the degree of temporal contiguity between the terms to be associated.[11]

Froeberg studied the effects of varying the length of the interval between the presentation of nonsense syllables in pairs, the interval being occupied by the repetition of two-digit numbers. Commenting on his results he says:

There is thus a distinct, though irregular, decrease in the number of right responses as the interval between the numbers of the pair in successive presentation increases.[12]

This view of the effect of the degree of temporal contiguity upon the speed of association is in accord with the close relation already observed between the speed of reaction of our subjects and the quickness of their discovery of the principle of drawing opposites. Moreover, the operation of such contiguity factors is evident in the fact that subjects commonly learned to recognize the critical status of numbers some time before they were able to give a reason for it.

Thus in dealing with novel situations through the abstraction of essential situation- and response-elements and the organization of these elements into serviceable concepts, mere frequency of repetition appears to play a rôle of no less importance than in the selection and combination of essential elements of stimulus and response into effective sensori-motor co-ordinations. The mechanical operation of frequency and of the factors mentioned in (a) and (b) above appears also to be quite as effective in determining the order of mastery of the elements in this type of conceptual learning as it has been supposed to be in sensori-motor learning.[13]

[11] Carr, H. A. "The Alternation Problem," Jour. Animal Behav., 1917, vol. 7, No. 5, p. 384.

[12] Froeberg, Sven. "Simultaneous versus Successive Association," Psych. Rev., 1918, vol. 25, No. 2, p. 162.

[13] Under the conditions of maze learning the process mentioned in (a) need involve only direct responses of the conditioned-reflex type to present stimuli. This statement would appear to imply the operation of "retroactive association," the possibility of which has been questioned by Hubbert and Lashley (*op. cit.*) and by Peterson (Joseph Peterson, "Frequency and Re-

B. *Combination of Essential Elements.*—In the early stages of the analysis of our problems, as at the beginning of the learning of telegraphy, typewriting, chess playing, etc., the individual elements are apprehended in relative isolation. Attention to one element precludes attention to others at the moment, with the result that the responses are of a random, haphazard character. With the progress of learning the span of attention is gradually broadened by the combination and organization of individual elements into larger, more meaningful units through which it is possible to focus one's relevant experiences upon new situations with a minimum loss of time and effort.

Regarding this process of combination and organization of essential elements Cleveland says:

> How organization can best be brought about is still an open question. . . . Its ultimate nature we do not know. To a great extent the material organizes itself, i.e., the organization is physiological and a matter of growth.[14]

That this organization is very largely independent of explicit conscious direction is indicated by the fact that the critical status of numbers was usually recognized before the subject was able to give a reason for it; that the prinicple of drawing opposites

cency Factors in Maze Learning by White Rats," Jour. Animal Behav., 1917, vol. 7, No. 5) though it is not entirely clear what connotation is to be given the term. If a "pleasurable situation" at the end of the series of responses to be learned is an essential factor, it is quite apparent that there was no retroactive association operative in the learning of our subjects. But the inclusion of this factor was seemingly not intended—though one cannot but suspect that the aversion to the idea of retroactive association arises from its early complication with the controverted stamping-in effects of pleasure—since, after its definition in terms of reactions leading up to a pleasurable situation, the process is restated in terms of conditioned reflexes.

Hubbert and Lashley's evidence against the presence of retroactive association in maze learning is, to say the least, inconclusive. The effect of the marked progressive shortening of the true runways toward the center of the maze was not determined. Peterson (Joseph, *op. cit.*, p. 361) has pointed out two other sources of weakness in their argument. Yet he writes (p. 362): "There seems to be no 'retroactive association' necessary, as Hubbert and Lashley rightly conclude." His evidence for this more conservative view is, however, drawn from his own experiments.

[14] Cleveland, A. A. "The Psychology of Chess and of Learning to Play It," American Journal of Psychology, 1907, vol. XVIII, p. 297.

was often followed and sometimes clearly formulated before the reason for it was known, and that solutions for series sometimes obtruded themselves upon a subject's consciousness after a period of rest during which no thought had been given to the problem and prior to which the subject's grasp of the elements of the problem had been characterized by confusion and lack of organization. Moreover, the unreasoned recognition of the principle of drawing opposites, and the backward order of learning of critical numbers, are clearly explicable on the basis of frequency of repetition and close temporal contiguity of the elements which were associated, as already shown.

The essential elements were unified not so much by direct association from one element to another as by the association of each element with a common intervening symbol. The character and manner of selection of such a symbol is a matter of some importance. Sometimes, during the early stages of unification, the intervening symbol is merely one of the elements of the concrete situation, or a direct response to such a situation-element, which happens to occur more frequently than others or to possess some other advantage, perhaps of intensity, duration, or position. Thus the goal of the trial, or the attitude aroused by defeat, or the verbal expression of the latter appeared to serve as a common symbol in the early stages of analysis and organization.

But as soon as enough numbers had been classified as "losing," "impossible," or "insoluble" to give the subject a general schematic impression of the character of the problem, the efforts at organization began to proceed more largely through the arousal and application of old concepts which functioned as a crude sort of hypotheses. Here the associations between the elements and the symbol were already in existence, provided the conception, or hypothesis, was correct. Henceforth the problem of unification became more a matter of calling up and testing out of various trial hypotheses than of the formation of new associations.

The various concepts which, under the influence of the general conception of the problem, functioned as hypotheses, appear to have been aroused in three fairly distinct ways: (a) through

their association with a specific element of the problem, (b) through the combined associative pull of a number of disconnected elements observed in rapid succession, and (c) through their association with *relations* observed to exist between individual elements, or, perhaps, with symbols representing such relations.

As an Example of (a), a subject has conceived of 9 as an insoluble number. It occurs to him that 9 is an odd number or that it is a multiple of 3, and he straightway infers that all odd numbers or all multiples of 3, as the case may be, are insoluble. This is the all but exclusive manner of origination of hypotheses in the early stages of learning when, owing to a lack of acquaintance with the elements of the problem, the span of attention to the elements is very narrowly limited. Subjects who have passed beyond the early stages of learning also revert to this form under the stress of emotion, self-consciousness, or fatigue, which tend seriously to narrow the span of attention to the elements of the problem. There are some individuals who, through native limitations of the span of attention seem doomed to a marked dependence upon this primitive form of arousal of hypotheses. These subjects were prone to react hastily. They drew rapidly and, disregarding recently observed and often well-known facts, frequently jumped to unwarranted conclusions from which they showed little ability to extricate themselves by crucial tests. That the limitation of the span of attention exhibiting itself in this lack of inhibition is probably native is indicated by the fact that the reactions of these subjects continued to be relatively rapid and ineffective even after an acquaintance had been gained with a fairly large number of elements. There are, of course, some subjects who are able to deal both rapidly and effectively with a wide variety of elements. Subjects iii and xi were the most notable examples of this type.

The origination of hypotheses in the manner described in (b) was observed in the later stages of learning, particularly in the work of subjects who inhibited the primitive tendency to generalize upon every instance and were intent upon the discovery and retention of several critical numbers prior to generalization.

With repeated recall of such previously observed critical numbers there was an increased facility manifesting itself in greater speed of repetition. Finally, after one or two repetitions of several critical numbers in rapid succession, the subject would suddenly announce his belief that all multiples of 3 are insoluble. Likewise, the later discovery of the superficial relation of critical numbers to the sum of the L- and H-draws often appeared to arise from the combined associative pull of a number of rapidly reviewed instances the essential feature of which as separate instances were not clearly apprehended. Here also, as in dealing with the critical numbers of a single series, the first general formulation was usually given in a tentative manner and further evidence sought for its proof. But the further evidence by which proof was established was usually of exactly the same type as that which had preceded the formulation, namely, the enumeration of more confirmatory instances and the absence of refractory ones.[15] Sometimes hypotheses originated in explicit analysis and comparison of the elements of one or more series. This is the type of origin mentioned in (c) above. At its best this process involves a clear insight into the causal relations of the phenomena in question and carries its own confirmation. At its worst it results in false analogies of the sort mentioned in the discussion of negative transfer (see above, p. 95 ff.). This process is dependent upon a fair acquaintance with the elements of the problem and is therefore impossible in the early stages of adjustment to really novel situations.

But the function of incipient hypotheses was not confined to the mere unification of previously discovered elements. These trial conceptions served also to direct attention to hitherto unknown elements and so aided materially in the discovery of new data.[16] The latter function is particularly noticeable in case of such concepts as even and odd numbers, every other number, multiples of 3, etc. The verification of such an hypothesis through its appliation to previously known data and its functioning in the discovery of new facts are but slightly different phases

[15] *Cf.* Dewey, "Studies in Logical Theory," p. 174.
[16] *Cf.* Dewey, *op. cit.*, p. 145.

of the same process of application. Both can be explained upon the basis of the laws of habit.[17] For example, suppose that 15 is under consideration and is conceived as a multiple of 3. Other multiples of 3 are immediately called to mind by association. But there is no invariable sequence. Sometimes, in accordance with the habit of counting upwards, 18 and 21 are the first associates aroused; sometimes the first associates to be aroused are 12 and 9, owing perhaps to the recency of their repetition. Simple associations of this sort probably account for much of the directive value of the more elaborate hypotheses of science.

There is a notable difference in the explicitness and stability of hypotheses in different stages of their development. Incipient hypotheses often occur in the early stages of the work as fleeting and perhaps vague insights. Further familiarization with the elements of the problem gives greater explicitness and stability to these hypotheses, though lapses still occur as is evidenced by the subjects' failure to make even the simplest and most obvious applications. A degree of instability sometimes persists long into the later stages of the experiment owing to the rivalry of conflicting or superfluous hypotheses which, though entertained earlier in the game, may have been neglected in the meantime. Book found a similar instability in the building up of co-ordinations in typewriting. He says:

It was observed by the learners that the older and more elemental habits used in the earlier stages of writing tended strongly to persist and force themselves upon the learner long after they had been superceded by higher-order habits. At every lapse in attention or relaxation of effort, the older habits stepped forward, as it were, and assumed control, thereby tending to perpetuate themselves. Only when a high degree of effort was being permanently applied . . . was attention forced to lay hold of the higher and more economical methods of work." Referring later to these fluctuations of attention and performance he says: "They were wholly beyond the learner's control. He

[17] This is in accord with Thorndike's view that "learning by inference is not opposed to, or independent of, the laws of habit, but is really their necessary result under the conditions imposed by man's nature and training." (Thorndike, E. L. "Educational Psychology," vol. II, p. 36.)

could not avoid them and could do little to regulate or control them."[18]

Continued attempts to deal with similar materials brought about a gradual strengthening of the associations necessary for the more ready arousal of appropriate concepts and the consequent failure of inappropriate ones to suggest themselves. This gradual automatization of the higher conceptual responses was observed by Cleveland in his study of chess. After giving an account of the combination of significant elements into larger complexes in which general terms "built up step by step" became increasingly more prominent, he says:

We are in the habit of speaking of the automatic in the motor realm, meaning by it that certain movements or combinations of movements are carried on without conscious guidance. Is there such a thing as automatism in the realm of the purely intellectual? It seems to me that this question is to be answered in the affirmative. There is something in the purely intellectual life corresponding to motor automatism, which is shown in the ability to think symbolically or abstractly, and thus to handle large masses of detail with a minimum of conscious effort. It involves the increasing ability to take in during a single pulse of attention a larger and larger group of details which means, of course, that the attention is no longer needed for each one.[19]

The combination of elements entering into a concept through various stages of automatization, is well illustrated in Fisher's study already referred to. Her subjects were presented with a series of ten somewhat similar figures exposed in succession for three seconds each. Each group was given a nonsense name, and the task consisted in defining the group name after the observation of the series. When repeated exposures revealed no additional essential features the exposures were discontinued; but the subjects continued at later sittings to recall the essential features of each group and report them as usual. Regarding the automatization of these concepts the author says:

Functionally viewed (i.e., regarded from the point of view of the difficulty and effortfulness, or the ease and mechanizedness

[18] Book, "The Psychology of Skill," pp. 94, 122.

[19] Cleveland, "The Psychology of Chess and Learning to Play It," American Journal of Psychology, vol. 18, p. 300.

with which the concept-meanings entered consciousness), the recalls ranged from an initial form in which more or less hesitation was present, to a final form which was marked by a high degree of mechanization and where the spoken statements followed in uneventful fashion, either immediately upon the instructions themselves, or upon a brief and transitory visual or verbal image which served to 'set off' the train." Describing the final stage of the developments of the concepts under observation she continues: "At this stage the experience of generality was nothing more than an unhesitating, ready, and even mechanical mentioning of the general features. . . . The generality experience was based essentially upon nothing more than a highly mechanized association between the words 'Zalofs are objects having,'—or 'Zalofs always have,'—and the enumeration of the essentials. The recalls were often given on a very automatic fashion."

In recapitulation it may be said that, in the early stages of learning, those elements which occurred most frequently and in closest temporal contiguity were generally the first to be combined into higher units. Certain situation- and response-elements which occurred most often and in closest proximity to other elements, became so closely associated with the latter as to serve as symbols through which the various elements were coordinated and their subsequent recall much facilitated. In the later stages of learning symbols representing well-organized concepts were called in by association. Through the medium of these symbols, and again by means of specific associations, meanings were transferred from previously known to newly discovered elements. Associations of this sort at first functioned slowly and imperfectly, but continued repetition brought about a facility and even automaticity of functioning comparable to that of sensori-motor co-ordinations.

C. *Application of Knowledge.*—As previously mentioned no marked tendency was observed for perceptual solutions to find application beyond the limits of the specific concrete situations in which they occurred. This absence of transfer was attributed to the fact that responses were made *directly* to the objective situation rather than indirectly through the intervention of a symbol. The solution of numbers beyond the range of direct

perceptual control showed, on the other hand, a very marked tendency to apply to new situations. This fact was attributed to the intervention of symbols, which was necessitated by the extent and complexity of the materials to be controlled in the attainment of a solution.

The reason for the superior applicability of solutions involving the use of symbols is not far to seek. The numerical symbols used in counting the beads remaining before each draw have been more frequently repeated than the concrete situations underlying the perceptual solutions and are therefore more easily recalled. Because of the greater ease of recall of often-repeated symbols than of seldom-observed situations, whatever meanings have been detached from the latter and associated with such symbols are more easily aroused in the presence of new situations. This advantage in the facility of recall of general symbols is important in as much as the conditions of application generally require the recall of the materials to be applied, while attention is directed primarily to the present problematic situation.

But it is not through the greater ease of recall of meanings which have grown out of the problem situation and become attached to the symbols so much as through the arousal of previously formed associations that such symbols serve to facilitate the application of old experiences to new situations. The symbols here employed in the reaction to all problems above 6 or 7 beads, were necessarily numerical symbols which were already organized into various orderly sequences. The moment such a symbol is used each of these sequences tends to be aroused through association, and thus to extend the meaning of the term under momentary consideration to the other terms of the suggested series. Thus 12, pronounced insoluble and conceived as an even number, tended strongly to suggest the insolubility of other even numbers such as 10 and 14. But, conceived as a multiple of 3, it tended to suggest the insolubility of 9, 15, etc. The tendency for meanings to apply themselves to new data through the medium of such general symbols was often so strong as to distort the memory of frequently observed facts.* Thus one of the preliminary subjects, observing the insolubility

of 18 beads in Series 1-2 and conceiving it as an even number, immediately lost his somewhat feeble grasp upon previously observed facts and "remembered" positively that 16, 14, 12, and 10 were also insoluble numbers.

The range of application is obviously determined largely by the generality of the symbol through which transfer[20] is to be effected, i.e., by the number and variety of *particulars* represented by the symbol. When, for example, the critical-number meanings were associated with 6, they were applicable to but one number in a single series. When these meanings became associated with the more general symbol, *a-multiple-of-3,* they became applicable to all numbers of the first series. Finally, when the still more general symbol, $L + H$, was substituted for the 3 of the earlier formula, the associated meanings became applicable to all problems of all continuous series. This is in accord with Judd's view that application is much facilitated by generalization.[21]

It is of course possible for knowledge to be expressed in general form without necessarily being generalized. There is, as Dewey observes, "always danger that symbols will not be truly representative; danger that instead of calling up the absent and remote in the way to make it enter a present experience, the linguistic media of representation will become an end in them-

* This tendency was probably strengthened in our experiments by the general conception of the problem.

Speaking of the influence of specific associations upon thinking, Thorndike says: "It has long been apparent that man's *erroneous* inferences—his unsuccessful responses to novel situations—are due to the action of misleading connections and analogies to which he is led by the laws of habit. It is also the fact, though it is not so apparent, that his successful responses are due to fruitful connections and analogies to which he is led by the same laws." (Educational Psychology, vol. II, p. 48.) The operation of the law of habit is here, however, as obvious in the arousal of correct as of incorrect responses through the medium of the general symbol.

[20] Bode argues that "transfer of training, when translated into terms of presentday knowledge, means the extension or applicaiton of meanings to new problems or new situations." (Boyd Henry Bode, "A Reinterpretation of Transfer of Training," Educational Administration and Supervision, vol. V, 1919, p. 107.)

[21] Judd, C. H. "Psychology of High-School Subjects," pp. 392-435.

selves."[22] Even though the symbol may previously have acquired a wide range of meaningful associations, it may, through the subject's lack of acquaintance with the elements of the present situation, function as a mere verbal response. Thus the hypothesis that that all multiples of 3 are insoluble often failed to function in any noticeable degree until long after its first explicit formulation, if such formulation chanced to occur before a fair acquaintance with the elements of the situation had been gained. That the degree of acquaintance with the elements strongly affects the applicability of generalizations to new problems, is also suggested by the fact that the degree of transfer from Series 1-2 to Series 1-3 was much greater for subjects whose progress through the first series was slow than for those who progressed rapidly.

The discovery of new elements of the situation through the associative action of a general symbol and the transfer of old meanings to these elements through the same medium, must inevitably lead to errors, since there is no certainty that the required symbol will be the first aroused or that when once aroused this symbol will call up only the correct association. Trial and error is obviously an essential method of procedure so long as any of the elements of the old or the new situation remain unknown, i.e., whenever new applications are being attempted. If the old situation is not fully analyzed, the generalization may be erroneous or inadequate; if analysis of the new situation is not complete, nonessential elements may dominate attention and thus bring about the arousal of inappropriate conceptions.

From the foregoing facts it is apparent that transfer is much facilitated by explicit generalization of experiences. This facilitation can be accounted for on the basis of the operation of specific associations between the symbols required for generalization and the elements of direct experience, which furnish the materials for generalization. The degree of transfer appears to depend upon both the number and the strength of such associations. It is also dependent largely upon the degree of acquaintance with the elements of the problematic situation to which ap-

[22] Dewey, J. "Democracy and Education," p. 272.

plication is to be affected. The educational value of a habit of analyzing and generalizing experiences, which may, according to Judd,[23] be developed by constant exercise of those functions, is beyond question. But it should not be forgotten that this habit must function in more or less familiar materials and through the operation of specific associations. Its value depends far more upon a gradual ordering of daily experiences, through repeated attempts at analysis and generalization, into such a flexible, systematic form as will make them available for future use, than upon sudden transformation of experiences at the moment of need.

D. *General Conclusions.*—Trial and error appears to be a universal method of procedure in learning of the problem-solving type. Not only does this procedure dominate the early stages of analysis of new materials, but it is a conspicuous factor in the determination of progress in the generalization of knowledge and its application to new situations. The field of variation is gradually limited through the effects of generalization, and the testing out of trial responses is facilitated through the familiarization of the elements of the situation. Thus larger and larger units of response come to be represented ideationally and tested out either overtly or in imagination; but the general trial-and-error character of the process remains always the same.

The most obvious factors in the selection and accentuation of essential elements were frequency of repetition of elements and their relative nearness to a goal, or end of action. Generalization and application of experiences were apparently somewhat less, though still largely, controlled by the same factors.

Progress in the detachment of elements and their generalization and application to new situations can be traced largely to the gradual formation and automatization of specific associations and to the associative arousal of previously formed concepts. But it should not be forgotten that old concepts wrung into service in this manner are themselves the product of earlier processes of gradual formation and mechanization of associations, essentially similar to the learning of sensori-motor co-ordina-

[23] *Op. cit.*, pp. 432-435.

tions. The learning process is not so much modified as abridged by this action of old concepts. The difference between sensori-motor learning and learning through abstraction and generalization is not so much a difference of method as of the type and complexity of previously established inter-relations of the materials to be organized. All of our data appear to confirm the view recently expressed by Thorndike that "Thinking and reasoning do not seem in any useful sense opposites of automatism, custom, or habit, but simply the action of habits in cases where the elements of the situation compete and co-operate notably."[24]

All ideas of theoretical importance which are expressed in the foregoing pages have been stated previously by other writers. The only originality which is claimed for the present study is to be found in the experimental verification of some of these ideas and in the development of a technique which, it is hoped, may prove to be of value in the further investigation of some of the problems in this field.

[24] Thorndike, E. L. "The Psychology of Thinking in the Case of Reading," Psychological Review, vol. XXIV, 1917, pp. 233-234.

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